ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a^2}{\sqrt{r_a}}\le\sum_{cyc}\frac{a^2}{\sqrt{h_a}}$$

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$$WLOG\ let's\ assume\ that\ a\ge b\ge c\ \Rightarrow$$

$$a^2\ge b^2\ge c^2\ \Rightarrow\frac{1}{\sqrt{r_a}}\le\frac{1}{\sqrt{r_b}}\le\frac{1}{\sqrt{r_c}}\ and\ \frac{1}{\sqrt{h_a}}\ge\frac{1}{\sqrt{h_b}}\ge\frac{1}{\sqrt{h_c}}$$

$$\frac{1}{\sqrt{h_a}}=\sqrt{\frac{\frac{1}{r_b}+\frac{1}{r_c}}{2}}\underbrace{\ge}_{Power\ Mean}\frac{1}{2}\left(\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}}\right)\Leftrightarrow\sum_{cyc}\frac{1}{\sqrt{r_a}}\le\sum_{cyc}\frac{1}{\sqrt{h_a}}$$

$$\sum_{cyc}\left(a^2\cdot\frac{1}{\sqrt{r_a}}\right)\underbrace{\le}_{Chebyshev}\frac{1}{3}\cdot\sum_{cyc}(a^2)\cdot\sum_{cyc}\left(\frac{1}{\sqrt{r_a}}\right)\le$$

$$\le\frac{1}{3}\cdot\sum_{cyc}(a^2)\cdot\sum_{cyc}\left(\frac{1}{\sqrt{h_a}}\right)\le\sum_{cyc}\left(a^2\cdot\frac{1}{\sqrt{h_a}}\right)$$

$$\therefore\sum_{cyc}\frac{a^2}{\sqrt{r_a}}\le\sum_{cyc}\frac{a^2}{\sqrt{h_a}}$$

$$Equality\ holds\ if\ and\ only\ if\ the\ triangle\ is\ equilateral.$$