ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{a^3}{r_a}}\le\sum_{cyc}\sqrt{\frac{a^3}{h_a}}$$

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$$WLOG\ let's\ assume\ that\ a\ge b\ge c\ \Rightarrow$$

$$\sqrt{a^3}\ge\sqrt{b^3}\ge\sqrt{c^3}\Rightarrow\frac{1}{\sqrt{r_a}}\le\frac{1}{\sqrt{r_b}}\le\frac{1}{\sqrt{r_c}}\ and\ \frac{1}{\sqrt{h_a}}\ge\frac{1}{\sqrt{h_b}}\ge\frac{1}{\sqrt{h_c}}.$$

$$\frac{1}{\sqrt{h_a}}=\sqrt{\frac{\frac{1}{r_b}+\frac{1}{r_c}}{2}}\overbrace{\ge}^{Power\ Mean}\frac{1}{2}\left(\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}}\right)\Leftrightarrow\sum_{cyc}\frac{1}{\sqrt{r_a}}\le\sum_{cyc}\frac{1}{\sqrt{h_a}}$$

$$\sum_{cyc}\left(\sqrt{a^3}\cdot\frac{1}{\sqrt{r_a}}\right)\overbrace{\le}^{Chebyshev}\frac{1}{3}\cdot\sum_{cyc}\left(\sqrt{a^3}\right)\cdot\sum_{cyc}\left(\frac{1}{\sqrt{r_a}}\right)\le$$

$$\le\frac{1}{3}\cdot\sum_{cyc}\left(\sqrt{a^3}\right)\cdot\sum_{cyc}\left(\frac{1}{\sqrt{h_a}}\right)\le\sum_{cyc}\left(\sqrt{a^3}\cdot\frac{1}{\sqrt{h_a}}\right)$$

$$\therefore\sum_{cyc}\sqrt{\frac{a^3}{r_a}}\le\sum_{cyc}\sqrt{\frac{a^3}{h_a}}$$

$$Equality\ holds\ if\ and\ only\ if\ the\ triangle\ is\ equilateral.$$