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In $\triangle ABC$ the following relationship holds:

$$\sum_{cyc} \frac{a}{\sqrt{r_a}} \leq \sum_{cyc} \frac{a}{\sqrt{h_a}}$$

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WLOG let's assume that $a \geq b \geq c \Rightarrow \frac{1}{\sqrt{r_a}} \leq \frac{1}{\sqrt{r_b}} \leq \frac{1}{\sqrt{r_c}}$ and $\frac{1}{\sqrt{h_a}} \geq \frac{1}{\sqrt{h_b}} \geq \frac{1}{\sqrt{h_c}}$.

$$\frac{1}{\sqrt{h_a}} = \sqrt{\frac{\frac{1}{r_b} + \frac{1}{r_c}}{2}} \stackrel{\text{Power Mean}}{\geq} \frac{1}{2} \left(\frac{1}{\sqrt{r_b}} + \frac{1}{\sqrt{r_c}} \right) \Leftrightarrow \sum_{cyc} \frac{1}{\sqrt{r_a}} \leq \sum_{cyc} \frac{1}{\sqrt{h_a}}$$

$$\sum_{cyc} \left(a \cdot \frac{1}{\sqrt{r_a}} \right) \stackrel{\text{Chebyshev}}{\geq} \frac{1}{3} \cdot \sum_{cyc} (a) \cdot \sum_{cyc} \left(\frac{1}{\sqrt{r_a}} \right) \leq \frac{1}{3} \cdot \sum_{cyc} (a) \cdot \sum_{cyc} \left(\frac{1}{\sqrt{h_a}} \right) \leq \sum_{cyc} \left(a \cdot \frac{1}{\sqrt{h_a}} \right)$$

$$\therefore \sum_{cyc} \frac{a}{\sqrt{r_a}} \leq \sum_{cyc} \frac{a}{\sqrt{h_a}}$$

Equality holds if and only if the triangle is equilateral.