ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc} \sin\frac{A}{2} \le \sqrt{2+\frac{R}{8r}}$$

Proposed by Neculai Stanciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\sin\frac{A}{2}\right)^2 = \sum_{cyc}\sin^2\frac{A}{2} + 2\sum_{cyc}\left(\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= \frac{2R-r}{2R} + \left(2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\left(\sum_{cyc}\operatorname{cosec}\frac{A}{2}\right)$$

$$= \frac{2R-r}{2R} + \frac{r}{2R}.\sum_{cyc}\sqrt{\frac{bc(s-a)}{r^2 s}} \overset{CBS}{\le} \frac{2R-r}{2R} + \frac{1}{2R.\sqrt{s}}.\sqrt{\sum_{cyc} ab}.\sqrt{\sum_{cyc}(s-a)}$$

$$= \frac{2R-r}{2R} + \frac{1}{2R.\sqrt{s}}.\sqrt{s^2+4Rr+r^2}.\sqrt{s} \overset{Gerretsen}{\le} \frac{2R-r}{2R} + \frac{1}{2R}.\sqrt{4R^2+8Rr+4r^2}$$

$$= \frac{2R-r}{2R} + \frac{2R+2r}{2R} = \frac{4R+r}{2R} \overset{?}{\le} 2+\frac{R}{8r} \Leftrightarrow R(R+16r) \overset{?}{\ge} 4r(4R+r) \Leftrightarrow R^2 \overset{?}{\ge} 4r^2$$

$$\rightarrow \text{true via Euler} \therefore \sum_{cyc}\sin\frac{A}{2} \le \sqrt{2+\frac{R}{8r}}\ \forall\ ABC,$$

$$'' = '' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

Solution 2 by Adgozel Adgozalov-Azerbaijan

In any triangle *ABC*, the following identity for half-angles is well-known:

$$\sum_{cyc}\sin\frac{A}{2} = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1+\frac{r}{R}$$

Therefore, the inequality is equivalent to:

$$\left(1+\frac{r}{R}\right)^2 \le 2+\frac{R}{8r}$$

Let $x=\frac{R}{r}$**. By Euler's inequality, *x*≥ 2. Substituting *x* gives:**

$$\left(1+\frac{1}{x}\right)^2 \leq 2+\frac{x}{8}$$

$$1+\frac{2}{x}+\frac{1}{x^2} \leq 2+\frac{x}{8} \iff \frac{x}{8}+1-\frac{2}{x}-\frac{1}{x^2} \geq 0$$

Multiplying by $8x^2 > 0$:

$x^3 + 8x^2 - 16x - 8 \geq 0$

Factoring the polynomial:

$(x-2)(x^2 + 10x + 4) \geq 0$

Since $x = \frac{R}{r} \geq 2$ we have $x-2 \geq 0$ and $x^2+10x+4 > 0$, so the inequality holds directly.
Equality holds if and only if $x = 2 \iff R = 2r$, which corresponds to an equilateral triangle.