ROMANIAN MATHEMATICAL MAGAZINE

In $any$ $\Delta$ ABC the $fo\ell\ell$owing $re\ell a$tionship $ho\ell ds$ :

$$\sum_{cyc}\frac{m_b - m_c}{(m_a + m_b)(2\sin^2 A + 2\sin^2 B - \sin^2 C)} \geq 0$$

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$$\left(\sum_{cyc}\frac{x}{y}\right)^2 \overset{?}{\geq} \frac{1}{xyz}\left(3xyz + \sum_{cyc}(xy(x+y))\right) = 3 + \sum_{cyc}\frac{x+y}{z}$$

$$= 3 + \sum_{cyc}\frac{y}{x} + \sum_{cyc}\frac{x}{y} \Leftrightarrow \sum_{cyc}\frac{x^2}{y^2} + \sum_{cyc}\frac{y}{x} \underset{①}{\overset{?}{\geq}} 3 + \sum_{cyc}\frac{x}{y}$$

$$\text{Now, } \sum_{cyc}\frac{x^2}{y^2} \geq \frac{1}{3}\left(\sum_{cyc}\frac{x}{y}\right)^2 \overset{AM-GM}{\geq} \frac{1}{3}.3.\sum_{cyc}\frac{x}{y} \Rightarrow \sum_{cyc}\frac{x^2}{y^2} \geq \sum_{cyc}\frac{x}{y} \text{ and } \because \sum_{cyc}\frac{y}{x} \overset{AM-GM}{\geq} 3$$

$$\therefore \sum_{cyc}\frac{x^2}{y^2} + \sum_{cyc}\frac{y}{x} \geq \sum_{cyc}\frac{x}{y} + 3 \Rightarrow ① \text{ is true}$$

$$\therefore \left(\sum_{cyc}\frac{x}{y}\right)^2 \geq \frac{1}{xyz}\left(3xyz + \sum_{cyc}(xy(x+y))\right) = \frac{1}{xyz}\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right) \text{ and with}$$

$$x \equiv a, y \equiv b, z \equiv c, \text{ we get : } \left(\sum_{cyc}\frac{a}{b}\right)^2 \geq \frac{1}{abc}.\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right) \to (m) \text{ and now,}$$

$$\text{we seek to prove : } \sum_{cyc}\frac{b-c}{c^2(a+b)} \overset{?}{\geq} 0 \Leftrightarrow \sum_{cyc}\frac{(a^2b^4 - a^2b^2c^2)(c+a)}{a^2b^2c^2(a+b)(b+c)(c+a)} \overset{?}{\geq} 0$$

$$\Leftrightarrow abc.\sum_{cyc}ab^3 + \sum_{cyc}a^3b^4 \boxed{\underset{(*)}{\overset{?}{\geq}}} 2a^2b^2c^2.\left(\sum_{cyc}a\right)$$

$$\text{Now, } \sum_{cyc}a^3b^4 = a^2b^2c^2.\sum_{cyc}\frac{ab^2}{c^2} = a^2b^2c^2.\sum_{cyc}\frac{\left(\frac{b}{c}\right)^2}{\frac{1}{a}} \overset{Bergstrom}{\geq} a^2b^2c^2.\frac{abc.\left(\sum_{cyc}\frac{a}{b}\right)^2}{\sum_{cyc}ab}$$

$$\overset{\text{via }(m)}{\geq} a^2b^2c^2.\frac{abc.\frac{1}{abc}.(\sum_{cyc}a)(\sum_{cyc}ab)}{\sum_{cyc}ab} \therefore \sum_{cyc}a^3b^4 \geq a^2b^2c^2.\left(\sum_{cyc}a\right) \to (i)$$

$$\text{and again, } abc.\sum_{cyc}ab^3 = a^2b^2c^2.\sum_{cyc}\frac{b^2}{c} \overset{Bergstrom}{\geq} a^2b^2c^2.\frac{(\sum_{cyc}a)^2}{\sum_{cyc}a}$$

$$\therefore abc.\sum_{cyc} ab^3 \geq a^2b^2c^2.\left(\sum_{cyc} a\right) \to \text{(ii) and so, (i) + (ii)} \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{b-c}{c^2(a+b)} \geq 0 \text{ \& implementing it on a triangle with sides } \frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3},$$

$$\text{we arrive at}: \sum_{cyc} \frac{\frac{2}{3}.(m_b - m_c)}{\frac{2}{27}.(4m_c^2)(m_a + m_b)} \geq 0 \Rightarrow \sum_{cyc} \frac{m_b - m_c}{(m_a + m_b)(2a^2 + 2b^2 - c^2)} \geq 0$$

$$\Rightarrow \sum_{cyc} \frac{m_b - m_c}{(m_a + m_b)\left(4R^2(2\sin^2 A + 2\sin^2 B - \sin^2 C)\right)} \text{ and so,}$$

$$\sum_{cyc} \frac{m_b - m_c}{(m_a + m_b)(2\sin^2 A + 2\sin^2 B - \sin^2 C)} \geq 0 \ \forall \text{ ABC},$$

$$'' = '' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$