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In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{h_a}{bc}\le\sum_{cyc}\frac{r_a}{bc}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$LHS=\sum_{cyc}\frac{h_a}{bc}=\sum_{cyc}\frac{\frac{2F}{a}}{bc}=\sum_{cyc}\frac{2F}{abc}=\frac{6F}{abc}=\frac{6F}{4RF}=\frac{3}{2R}$$

$$RHS=\frac{r_a}{bc}+\frac{r_b}{ac}+\frac{r_c}{ba}\overset{AM-GM}{\ge}3\sqrt[3]{\frac{r_ar_br_c}{(abc)^2}}=3\sqrt[3]{\frac{p^2r}{16F^2R^2}}=$$

$$=3\sqrt[3]{\frac{p^2r}{16p^2R^2r^2}}=3\sqrt[3]{\frac{1}{16rR^2}}\overset{Euler}{\ge}3\sqrt[3]{\frac{1}{8R^3}}=\frac{3}{2R}=LHS$$

$$So,\qquad RHS\ge LHS\quad(Proved)$$

$$Equality\ \ holds\ \ if:\quad a=b=c$$