ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\left(\frac{h_b}{h_a}+\frac{h_c}{h_b}+\frac{h_a}{h_c}\right)^2 \geq \frac{h_a+h_b+h_c}{r}$$

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$$\left(\sum_{cyc}\frac{x}{y}\right)^2 \overset{?}{\geq} \frac{1}{xyz}\left(3xyz+\sum_{cyc}(xy(x+y))\right) = 3+\sum_{cyc}\frac{x+y}{z}$$

$$= 3+\sum_{cyc}\frac{y}{x}+\sum_{cyc}\frac{x}{y} \Leftrightarrow \sum_{cyc}\frac{x^2}{y^2}+\sum_{cyc}\frac{y}{x} \underset{①}{\overset{?}{\geq}} 3+\sum_{cyc}\frac{x}{y}$$

Now, $\sum_{cyc}\frac{x^2}{y^2} \geq \frac{1}{3}\left(\sum_{cyc}\frac{x}{y}\right)^2 \overset{AM-GM}{\geq} \frac{1}{3}.3.\sum_{cyc}\frac{x}{y} \Rightarrow \sum_{cyc}\frac{x^2}{y^2} \geq \sum_{cyc}\frac{x}{y}$ and $\because \sum_{cyc}\frac{y}{x} \overset{AM-GM}{\geq} 3$

$$\therefore \sum_{cyc}\frac{x^2}{y^2}+\sum_{cyc}\frac{y}{x} \geq \sum_{cyc}\frac{x}{y}+3 \Rightarrow ① \text{ is true}$$

$\therefore \left(\sum_{cyc}\frac{x}{y}\right)^2 \geq \frac{1}{xyz}\left(3xyz+\sum_{cyc}(xy(x+y))\right) = \frac{1}{xyz}\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)$ and with

$$x \equiv h_b, y \equiv h_a, z \equiv h_c, \left(\frac{h_b}{h_a}+\frac{h_a}{h_c}+\frac{h_c}{h_b}\right)^2 \geq \frac{1}{h_a h_b h_c}\left(\sum_{cyc}h_a\right)\left(\sum_{cyc}h_a h_b\right)$$

$$= \frac{1}{s^2 r}\left(\sum_{cyc}h_a\right).s^2 = \frac{h_a+h_b+h_c}{r} \text{ and so, } \left(\frac{h_b}{h_a}+\frac{h_c}{h_b}+\frac{h_a}{h_c}\right)^2 \geq \frac{h_a+h_b+h_c}{r}$$

$\forall \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)