ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{27r}{s}\le\sum_{cyc}\frac{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{\sin A}\le\frac{27R^2}{4rs}$$

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Solution by Jenish Rijal-Nepal

$$\sum_{cyc}\frac{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{\sin A}=\sum_{cyc}\frac{\frac{s(s-b)}{ac}+\frac{s(s-c)}{ab}}{\frac{2rs}{bc}}=$$

$$=\sum_{cyc}\frac{s(s-b)\cdot b+s(s-c)\cdot c}{2ars}=\sum_{cyc}\frac{b(s-b)+c(s-c)}{2ar}$$

$$=\sum_{cyc}\frac{b(a+c-b)+c(a+b-c)}{4ar}=$$

$$=\sum_{cyc}\frac{ab+ca+2bc-(b^2+c^2)}{4ar}=\sum_{cyc}\frac{a(b+c)-(b-c)^2}{4ar}=$$

$$=\sum_{cyc}\left[\frac{(b+c)}{4r}-\frac{(b-c)^2}{4ar}\right]=\frac{s}{r}-\sum_{cyc}\frac{(b-c)^2}{4ar}\le\frac{s}{r}\quad\because\sum_{cyc}\frac{(b-c)^2}{4ar}\ge 0$$

$$\sum_{cyc}\frac{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{\sin A}\le\frac{s}{r}=\frac{(2s)^2}{4rs}\overset{Mitrinovic}{\le}\frac{\left(3\sqrt{3}R\right)^2}{4rs}=\frac{27R^2}{4rs}$$

$$\sum_{cyc}\frac{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{\sin A}\overset{AM-GM}{\ge}\sum_{cyc}\frac{2\cos\frac{B}{2}\cos\frac{C}{2}}{\sin A}=$$

$$=\sum_{cyc}\frac{2\cos\frac{B}{2}\cos\frac{C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}\overset{AM-GM}{\ge}3\sqrt[3]{\frac{8\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}}{8\prod\left(\sin\frac{A}{2}\cos\frac{A}{2}\right)}}=$$

$$3\sqrt[3]{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}=3\sqrt[3]{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}\overset{Jensen}{\ge}$$

$$\ge 3\sqrt[3]{\cot\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)}=3\sqrt[3]{3\sqrt{3}}=3\sqrt{3}=\frac{27}{3\sqrt{3}}$$

$$\sum_{cyc}\frac{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{\sin A}\ge\frac{27}{3\sqrt{3}}=27\cdot\frac{1}{3\sqrt{3}}\overset{Mitrinovic}{\ge}27\cdot\frac{r}{s}=\frac{27r}{s}$$

$$\therefore\ \frac{27r}{s}\le\sum_{cyc}\frac{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{\sin A}\le\frac{27R^2}{4rs}$$

Equality holds if and only if the triangle is equilateral.