ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\left(\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\right)^2 \geq \frac{r_a+r_b+r_c}{r}$$

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$$\left(\sum_{cyc}\frac{x}{y}\right)^2 \overset{?}{\geq} \frac{1}{xyz}\left(3xyz+\sum_{cyc}(xy(x+y))\right) = 3+\sum_{cyc}\frac{x+y}{z}$$

$$= 3+\sum_{cyc}\frac{y}{x}+\sum_{cyc}\frac{x}{y} \Leftrightarrow \sum_{cyc}\frac{x^2}{y^2}+\sum_{cyc}\frac{y}{x} \underset{①}{\overset{?}{\geq}} 3+\sum_{cyc}\frac{x}{y}$$

$$\text{Now, } \sum_{cyc}\frac{x^2}{y^2} \geq \frac{1}{3}\left(\sum_{cyc}\frac{x}{y}\right)^2 \overset{AM-GM}{\geq} \frac{1}{3}.3.\sum_{cyc}\frac{x}{y} \Rightarrow \sum_{cyc}\frac{x^2}{y^2} \geq \sum_{cyc}\frac{x}{y} \text{ and } \because \sum_{cyc}\frac{y}{x} \overset{AM-GM}{\geq} 3$$

$$\therefore \sum_{cyc}\frac{x^2}{y^2}+\sum_{cyc}\frac{y}{x} \geq \sum_{cyc}\frac{x}{y}+3 \Rightarrow ① \text{ is true}$$

$$\therefore \left(\sum_{cyc}\frac{x}{y}\right)^2 \geq \frac{1}{xyz}\left(3xyz+\sum_{cyc}(xy(x+y))\right) = \frac{1}{xyz}\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right) \text{ and with}$$

$$x \equiv r_b, y \equiv r_a, z \equiv r_c, \left(\frac{r_b}{r_a}+\frac{r_a}{r_c}+\frac{r_c}{r_b}\right)^2 \geq \frac{1}{r_a r_b r_c}\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}r_a r_b\right)$$

$$= \frac{1}{s^2 r}\left(\sum_{cyc}r_a\right).s^2 = \frac{r_a+r_b+r_c}{r} \text{ and so, } \left(\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\right)^2 \geq \frac{r_a+r_b+r_c}{r}$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)