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In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{1}{r_a}+\frac{1}{r_b}\right)\left(\sqrt{r_a}+\sqrt{r_b}\right)^2 \le \frac{12R}{r}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$WLOG: a \le b \le c \ ;\ r_a \le r_b \le r_c;\ \ \sqrt{r_a} \le \sqrt{r_b} \le \sqrt{r_c}\ ;\ \frac{1}{r_a} \ge \frac{1}{r_b} \ge \frac{1}{r_c}$$

$$\sum_{cyc}\left(\frac{1}{r_a}+\frac{1}{r_b}\right)\left(\sqrt{r_a}+\sqrt{r_b}\right)^2 \overset{Chebyshev}{\le} \frac{1}{3}\sum_{cyc}\left(\frac{1}{r_a}+\frac{1}{r_b}\right).\sum_{cyc}\left(\sqrt{r_a}+\sqrt{r_b}\right)^2 =$$

$$= \frac{1}{3}.2\left(\sum_{cyc}\frac{1}{r_a}\right).2\left(\sum_{cyc} r_a + \sum_{cyc}\sqrt{r_a.r_b}\right) \overset{A-G}{\le} \frac{4}{3r}\left((4R+r)+(4R+r)\right) \overset{Euler}{\le}$$

$$\le \frac{4}{3r}.\frac{9R}{2}.2 = \frac{12R}{r}$$

$$Equality\ \ holds\ \ for:\ \ a=b=c$$