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In $\triangle ABC$ the following relationship holds:

$$\left(\sum_{cyc} \frac{a^2}{b^2 + c^2} \right) \left(\sum_{cyc} \frac{rr_a}{bc} \right) \leq \frac{(2R - r)^2}{4Rr}$$

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$$\begin{aligned} \sum_{cyc} \frac{a^2}{b^2 + c^2} &\stackrel{AM-GM}{\geq} \frac{1}{2} \sum_{cyc} \frac{a^2}{bc} = \frac{1}{2abc} \sum_{cyc} a^3 = \frac{1}{8Rpr} \cdot 2(p^3 - 3pr^2 - 6Rpr) = \\ &= \frac{1}{4Rr} (p^2 - 3r^2 - 6Rr) \stackrel{Gerretsen}{\geq} \frac{1}{4Rr} (4R^2 + 4Rr + 3r^2 - 3r^2 - 6Rr) = \frac{2R - r}{2r} \quad (1) \end{aligned}$$

$$\sum_{cyc} \frac{rr_a}{bc} = r \sum_{cyc} \frac{r_a}{bc} = \frac{r}{abc} (ar_a + br_b + cr_c) = \frac{r}{4Rrp} \cdot 2p(2R - r) = \frac{2R - r}{2R} \quad (2)$$

From (1) and (2):

$$\left(\sum_{cyc} \frac{a^2}{b^2 + c^2} \right) \left(\sum_{cyc} \frac{rr_a}{bc} \right) \leq \frac{2R - r}{2r} \cdot \frac{2R - r}{2R} = \frac{(2R - r)^2}{4Rr}$$

Equality holds for : $a = b = c$