ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{3r^2}{2R^2} \le \sum \frac{rr_a}{b^2+c^2} \le \frac{2R-r}{8r}$$

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$$\sum_{cyc} \frac{rr_a}{b^2+c^2} \overset{AM-GM}{\le} \sum_{cyc} \frac{rr_a}{2bc} = \frac{r}{2}\sum_{cyc}\frac{r_a}{bc} =$$

$$= \frac{r}{2}\left(\frac{ar_a+br_b+cr_c}{abc}\right) = \frac{r.2p(2R-r)}{2.4pr.R} = \frac{2R-r}{4R} \overset{Euler}{\le} \frac{2R-r}{8r} \quad (RHS)$$

$$\sum \frac{rr_a}{b^2+c^2} = r\sum_{cyc}\frac{r_a}{b^2+c^2}$$

$$In\ \ \Delta ABC\ wlog\ : \begin{cases} a \le b \le c \to r_a \le r_b \le r_c \\ b^2+c^2 \ge a^2+c^2 \ge b^2+a^2 \to \dfrac{1}{b^2+c^2} \le \dfrac{1}{a^2+c^2} \le \dfrac{1}{b^2+a^2} \end{cases}$$

$$\sum_{cyc}\frac{r_a}{b^2+c^2} \overset{Chebyshev}{\ge} \frac{1}{3}\sum_{cyc} r_a . \sum_{cyc}\frac{1}{b^2+c^2} == \frac{1}{3}(4R+r).\sum_{cyc}\frac{1^2}{b^2+c^2} \overset{Bergstrom}{\ge}$$

$$\ge \frac{4R+r}{3}.\frac{(1+1+1)^2}{2(b^2+c^2+a^2)} = \frac{4R+r}{3}.\frac{9}{2(b^2+c^2+a^2)} \overset{\substack{Euler\\Leibniz}}{\ge} \frac{9r}{3}.\frac{9}{2.9R^2} = \frac{3r^2}{2R^2}$$

$$\sum \frac{rr_a}{b^2+c^2} \ge \frac{3r^2}{2R^2} \quad (LHS)$$

$$Equality\ \ holds\ \ for: \quad a=b=c$$