ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{1}{r} \le \frac{r_a}{h_a^2} + \frac{r_b}{h_b^2} + \frac{r_c}{h_c^2} \le \frac{27R^3(R-r)}{8r^3s^2}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc} \frac{r_a}{h_a^2} = \sum_{cyc} \frac{s.\tan\left(\frac{A}{2}\right)}{\frac{4F^2}{a^2}} = \frac{s}{4F^2}\sum_{cyc} a^2\tan\left(\frac{A}{2}\right) = \frac{s}{4F^2}\sum_{cyc} a\cdot a\tan\left(\frac{A}{2}\right) =$$

$$= \frac{s}{4F^2}\sum_{cyc} a(r_a - r) = \frac{s}{4F^2}\left(\sum_{cyc} ar_a - \sum_{cyc} ar\right) =$$

$$= \frac{s}{4F^2}(2s(2R-r) - 2sr) = \frac{s}{4F^2}(4sR - 4sr) =$$

$$= \frac{s^2(R-r)}{F^2} \overset{Mitrinovici}{\le} \frac{\frac{27R^2}{4}(R-r)}{S^2r^2} = \frac{27R^2(R-r)}{4S^2r^2} \overset{Euler}{\le} \frac{27R^3(R-r)}{8S^2r^3} \quad (RHS)$$

$$\frac{r_a}{h_a^2} + \frac{r_b}{h_b^2} + \frac{r_c}{h_c^2} = \frac{\left(\frac{1}{h_a}\right)^2}{\frac{1}{r_a}} + \frac{\left(\frac{1}{h_b}\right)^2}{\frac{1}{r_b}} + \frac{\left(\frac{1}{h_c}\right)^2}{\frac{1}{r_c}} \overset{Bergstrom}{\ge} \frac{\left(\sum_{cyc}\frac{1}{h_a}\right)^2}{\sum_{cyc}\frac{1}{r_a}} = \frac{\frac{1}{r^2}}{\frac{1}{r}} = \frac{1}{r} \quad (LHS)$$

$$Equality\ \ holds\ \ for: \quad a = b = c$$