ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC and $\forall$ m, n $\geq$ 1, the following relationship holds :

$$\sum_{cyc}\frac{a^n+b^n}{h_c^m}\le\sum_{cyc}\frac{a^n+b^n}{r_c^m}$$

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WLOG we may assume $a\ge b\ge c$ and then : $b^n+c^n\le c^n+a^n\le a^n+b^n$ $(\because n\ge 1>0)$ and $\frac{1}{h_a^m}\ge\frac{1}{h_b^m}\ge\frac{1}{h_c^m}$ $(\because m\ge 1>0)$ and also, $\frac{1}{r_a^m}\le\frac{1}{r_b^m}\le\frac{1}{r_c^m}$

$$\left(\because\frac{1}{r_a}=\frac{1}{s\tan\frac{A}{2}}\le\frac{1}{r_b}=\frac{1}{s\tan\frac{B}{2}}\le\frac{1}{r_c}=\frac{1}{s\tan\frac{C}{2}}\text{ and }\because m\ge 1>0\right)\text{ and so,}$$

$$\sum_{cyc}\frac{b^n+c^n}{h_a^m}\overset{\text{Chebyshev}}{\le}\frac{1}{3}\left(\sum_{cyc}(b^n+c^n)\right)\left(\sum_{cyc}\frac{1}{h_a^m}\right)\to (1)\text{ and also,}$$

$$\sum_{cyc}\frac{b^n+c^n}{r_a^m}\overset{\text{Chebyshev}}{\ge}\frac{1}{3}\left(\sum_{cyc}(b^n+c^n)\right)\left(\sum_{cyc}\frac{1}{r_a^m}\right)\to (2)$$

$$\therefore (1)\text{ and }(2)\Rightarrow\text{it suffices to prove : }\sum_{cyc}\frac{1}{h_a^m}\overset{?}{\le}\sum_{cyc}\frac{1}{r_a^m}\Leftrightarrow 2^m.\sum_{cyc}(s-a)^m\overset{?}{\ge}\sum_{cyc}a^m$$

$$\Leftrightarrow 2^m.\sum_{cyc}x^m\ \boxed{\overset{?}{\ge}(*)}\ \sum_{cyc}(y+z)^m$$

$(x=s-a, y=s-b, z=s-c$ and then : $a=y+z, b=z+x, c=x+y)$

We note that : for m = 1, LHS of $(*)=2\sum_{cyc}x$ and RHS of $(*)=$ $\sum_{cyc}(y+z)=2\sum_{cyc}x=$ LHS of $(*)$ and when : m > 1, we have :

via Holder's Inequality, $y^m+z^m\ge\frac{1}{2^{m-1}}.(y+z)^m$ and analogs

$$\Rightarrow\sum_{cyc}(y^m+z^m)\ge\frac{1}{2^{m-1}}.\sum_{cyc}(y+z)^m\Rightarrow 2^m.\sum_{cyc}x^m\ge\sum_{cyc}(y+z)^m$$

$$\Rightarrow (*)\text{ is true and so, }\sum_{cyc}\frac{a^n+b^n}{h_c^m}\le\sum_{cyc}\frac{a^n+b^n}{r_c^m}\ \forall\,\Delta\text{ ABC and }\forall\text{ m, n}\ge 1,$$

" = " iff Δ ABC is equilateral (QED)