

ROMANIAN MATHEMATICAL MAGAZINE

In $\triangle ABC$ the following relationship holds:

$$\sum_{cyc} \frac{(2r_a^2 + r_b^2 + r_c^2)^2}{(m_b^2 + m_c^2)(r_b r_c + r_a r)} \geq \frac{2R^2}{r^2} + \frac{r_a + r_b + r_c}{r} + 1 + \frac{\sum_{cyc} (a-b)^2}{2r^2} + \frac{\sum_{cyc} (m_a - m_b)^2}{6r^2}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Nawar Alasadi-Iraq

$$\begin{aligned} r_b r_c + r_a r &= s(s-a) + (s-b)(s-c) \\ &= 2s^2 - s(a+b+c) + bc = 2s^2 - 2s^2 + bc = bc \end{aligned}$$

By the Cauchy-Schwarz inequality:

$$\sum_{cyc} \frac{(2r_a^2 + r_b^2 + r_c^2)^2}{(m_b^2 + m_c^2)bc} \geq \frac{(\sum_{cyc} (2r_a^2 + r_b^2 + r_c^2))^2}{\sum_{cyc} bc(m_b^2 + m_c^2)} = \frac{16(r_a^2 + r_b^2 + r_c^2)^2}{\sum_{cyc} bc(m_b^2 + m_c^2)}$$

We evaluate the denominator:

$$\begin{aligned} m_b^2 + m_c^2 &= \frac{4a^2 + b^2 + c^2}{4} \\ \sum_{cyc} bc(m_b^2 + m_c^2) &= \frac{1}{4} \sum_{cyc} bc(4a^2 + b^2 + c^2) = abc(a+b+c) + \frac{1}{4} \sum_{cyc} bc(b^2 + c^2) \\ &= 8Rrs^2 + \frac{1}{4} [2(s^2 + r^2 + 4Rr)(s^2 - r^2 - 4Rr) - 8Rrs^2] \\ &= \frac{1}{2} s^4 + 6Rrs^2 - \frac{1}{2} (r^2 + 4Rr)^2 \end{aligned}$$

The Right Hand Side (RHS) simplifies to:

$$\begin{aligned} \text{RHS} &= \frac{2R^2}{r^2} + \frac{4R+r}{r} + 1 + \frac{2s^2 - 24Rr - 6r^2}{2r^2} + \frac{2\sum_{cyc} m_a^2 - 2\sum_{cyc} m_a m_b}{6r^2} \\ &= \frac{2R^2}{r^2} + \frac{4R}{r} + 2 + \frac{s^2 - 12Rr - 3r^2}{r^2} + \frac{\sum_{cyc} m_a^2 - \sum_{cyc} m_a m_b}{3r^2} \end{aligned}$$

Using the fundamental triangle inequalities and

$$\sum_{cyc} r_a^2 = (4R+r)^2 - 2s^2, \text{ we obtain:}$$

$$\frac{16(r_a^2 + r_b^2 + r_c^2)^2}{\frac{1}{2} s^4 + 6Rrs^2 - \frac{1}{2} (r^2 + 4Rr)^2} \geq \text{RHS}$$