

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$\frac{R}{r} \cdot \sum_{cyc} \sqrt{\frac{m_a}{h_a}} + \sqrt[4]{\frac{1}{r^2} \sum_{cyc} r_a r_b} \geq \frac{3}{2} \cdot \left(1 + \sum_{cyc} \frac{r_a}{h_a} \right) + \sqrt[4]{16 \cdot \frac{R}{r} - 5} + \sum_{cyc} \left(\frac{m_a - h_a}{m_a} \right)^2$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \frac{3}{2} \cdot \left(1 + \sum_{cyc} \frac{r_a}{h_a} \right) &= \frac{3}{2} \cdot \left(1 + \sum_{cyc} \frac{s \tan \frac{A}{2} \cdot 4R \cdot \cos \frac{A}{2} \sin \frac{A}{2}}{2rs} \right) \\ &= \frac{3}{2} \cdot \left(1 + \frac{2R}{r} \cdot \sum_{cyc} \sin^2 \frac{A}{2} \right) = \frac{3}{2} \cdot \left(1 + \frac{2R}{r} \cdot \frac{2R - r}{2R} \right) = \frac{3R}{r} \\ \therefore \frac{R}{r} \cdot \sum_{cyc} \sqrt{\frac{m_a}{h_a}} - \frac{3}{2} \cdot \left(1 + \sum_{cyc} \frac{r_a}{h_a} \right) &= \frac{R}{r} \cdot \sum_{cyc} \left(\sqrt{\frac{m_a}{h_a}} - 1 \right) = \frac{R}{r} \cdot \sum_{cyc} \frac{m_a - h_a}{\sqrt{h_a} \cdot (\sqrt{m_a} + \sqrt{h_a})} \\ &\geq 2 \cdot \sum_{cyc} \frac{m_a - h_a}{m_a + h_a} \left(\because h_a \leq m_a \text{ and analogs and } \frac{R}{r} \stackrel{\text{Euler}}{\geq} 2 \right) \\ \therefore \frac{R}{r} \cdot \sum_{cyc} \sqrt{\frac{m_a}{h_a}} - \frac{3}{2} \cdot \left(1 + \sum_{cyc} \frac{r_a}{h_a} \right) &\geq 2 \cdot \sum_{cyc} \frac{m_a - h_a}{m_a + h_a} \rightarrow \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Now, } 2 \cdot \frac{m_a - h_a}{m_a + h_a} &\stackrel{?}{\geq} \left(\frac{m_a - h_a}{m_a} \right)^2 \Leftrightarrow 2m_a^2 \stackrel{?}{\geq} (m_a - h_a)(m_a + h_a) \left(\because m_a - h_a \geq 0 \right) \\ &\Leftrightarrow 2m_a^2 \stackrel{?}{\geq} m_a^2 - h_a^2 \Leftrightarrow m_a^2 + h_a^2 \stackrel{?}{\geq} 0 \rightarrow \text{true (strict inequality)} \\ \therefore 2 \cdot \frac{m_a - h_a}{m_a + h_a} &\stackrel{?}{\geq} \left(\frac{m_a - h_a}{m_a} \right)^2 \text{ \& analogs } \Rightarrow 2 \cdot \sum_{cyc} \frac{m_a - h_a}{m_a + h_a} \geq \sum_{cyc} \left(\frac{m_a - h_a}{m_a} \right)^2 \rightarrow \textcircled{2} \end{aligned}$$

$$\therefore \textcircled{1} \text{ and } \textcircled{2} \Rightarrow \frac{R}{r} \cdot \sum_{cyc} \sqrt{\frac{m_a}{h_a}} \stackrel{(*)}{\geq} \frac{3}{2} \cdot \left(1 + \sum_{cyc} \frac{r_a}{h_a} \right) + \sum_{cyc} \left(\frac{m_a - h_a}{m_a} \right)^2$$

$$\text{Again, } \sqrt[4]{\frac{1}{r^2} \sum_{cyc} r_a r_b} = \sqrt[4]{\frac{s^2}{r^2}} \stackrel{\text{Gerretsen}}{\geq} \sqrt[4]{\frac{16Rr - 5r^2}{r^2}} \therefore \sqrt[4]{\frac{1}{r^2} \sum_{cyc} r_a r_b} \stackrel{(**)}{\geq} \sqrt[4]{16 \cdot \frac{R}{r} - 5}$$

$$\text{So, } (*) + (**) \Rightarrow \frac{R}{r} \cdot \sum_{cyc} \sqrt{\frac{m_a}{h_a}} + \sqrt[4]{\frac{1}{r^2} \sum_{cyc} r_a r_b} \geq$$

$$\frac{3}{2} \cdot \left(1 + \sum_{cyc} \frac{r_a}{h_a} \right) + \sqrt[4]{16 \cdot \frac{R}{r} - 5} + \sum_{cyc} \left(\frac{m_a - h_a}{m_a} \right)^2 \quad \forall \Delta ABC,$$

" = " iff ΔABC is equilateral (QED)