

ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC following relationship holds:

$$\left(\frac{R}{r}\right)^2 \sum \sqrt{\frac{m_a^5}{w_a^3 h_a^2}} + \frac{1}{3r^2} \sum r_a r_b \geq 5 \sum \frac{r_a}{h_a} + 2 \sqrt[3]{\frac{32R}{r} - 37} + 16 \sum \left(\frac{m_a - w_a}{m_a + h_a}\right)^2$$

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$$\sum \sqrt{\frac{m_a^5}{w_a^3 h_a^2}} \stackrel{m_a \geq w_a \geq h_a}{\geq} \sum 1 = 3 \quad (1)$$

$$\frac{1}{3r^2} \sum r_a r_b = \frac{1}{3} \frac{s^2}{r^2} \stackrel{\text{Gerrestn}}{\geq} \frac{16Rr - 5r^2}{3r^2} \stackrel{\frac{R}{r} = x \geq 2}{=} \frac{1}{3} (16x - 5) \quad (2)$$

$$5 \sum \frac{r_a}{h_a} = 5 \sum \frac{\frac{F}{s-a}}{\frac{2F}{a}} = \frac{5}{2} \sum \frac{a}{s-a} = \frac{5}{2} \left(\frac{4R}{r} - 2 \right) = 5(2x - 1) \quad (3)$$

$$\begin{aligned} 16 \sum \left(\frac{m_a - w_a}{m_a + h_a} \right)^2 &\stackrel{w_a \geq h_a}{\geq} 16 \sum \left(\frac{m_a - h_a}{m_a + h_a} \right)^2 = \\ &= 16 \sum \left(\frac{\frac{m_a}{h_a} - 1}{\frac{m_a}{h_a} + 1} \right)^2 \stackrel{\text{Panatipal } m_a \geq h_a}{\geq} 16 \sum \left(\frac{\frac{R}{2r} - 1}{1 + 1} \right)^2 = \\ &= \sum \left(\frac{R}{r} - 2 \right)^2 = \sum (x - 2)^2 = 3(x - 2)^2 \quad (4) \end{aligned}$$

We need to show:

$$\left(\frac{R}{r}\right)^2 \sum \sqrt{\frac{m_a^5}{w_a^3 h_a^2}} + \frac{1}{3r^2} \sum r_a r_b \geq 5 \sum \frac{r_a}{h_a} + 2 \sqrt[3]{\frac{32R}{r} - 37} + 16 \sum \left(\frac{m_a - w_a}{m_a + h_a}\right)^2$$

Using (1), (2), (3), (4) we get:

$$3x^2 + \frac{1}{3} (16x - 5) \geq 5(2x - 1) + 2 \sqrt[3]{32x - 37} + 3(x - 2)^2$$

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$$9x^2 + 16x - 5 \geq 9x^2 - 6x + 21 + 6\sqrt[3]{32x - 37} \text{ or } 11x - 13 \geq 3\sqrt[3]{32x - 37} \text{ (A)}$$

$$\text{or let } y = \sqrt[3]{32x - 37} \text{ then } y \stackrel{x \geq 2}{\geq} \sqrt[3]{32 \times 2 - 37} = \sqrt[3]{27} = 3 \text{ \& } x = \frac{y^3 + 37}{32}$$

$$\text{From (A) we have } 11\left(\frac{y^3 + 37}{32}\right) - 13 \geq 3y \text{ or } 11y^3 - 96y - 9 \geq 0$$

$$\text{or } (y - 3)(11y^2 + 33y + 3) \geq 0 \text{ true as } y \geq 3$$

Equality holds for an equilateral triangle.