

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$\left(\frac{R}{2r}\right)^3 \cdot \sum_{cyc} \frac{r_a^3 + r_a r_b r_c}{w_a (h_b^2 + h_c^2)} \geq \sum_{cyc} \left(\sqrt{\frac{m_a}{h_a}} \cdot \frac{r_b r_c}{w_b w_c} \right)$$

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$$\begin{aligned} (b+c)^2 &\stackrel{?}{\geq} 32Rr \cdot \cos^2 \frac{A}{2} = 8r(r_b + r_c) = 8r^2 s \left(\frac{1}{s-b} + \frac{1}{s-c} \right) \\ &= 8(s-a)(s-b)(s-c) \cdot \frac{a}{(s-b)(s-c)} = 4a(b+c-a) \\ \Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) &\stackrel{?}{\geq} 0 \Leftrightarrow (b+c-2a)^2 \stackrel{?}{\geq} 0 \rightarrow \text{true} \\ \therefore b+c &\geq \sqrt{32Rr} \cdot \cos \frac{A}{2} \text{ and analogs } \Rightarrow \frac{w_a}{h_a} = \frac{2R \cdot 2bc \cdot \cos \frac{A}{2}}{(b+c)bc} \leq \frac{4R \cdot \cos \frac{A}{2}}{\sqrt{32Rr} \cdot \cos \frac{A}{2}} = \sqrt{\frac{R}{2r}} \\ \Rightarrow w_a(h_b^2 + h_c^2) &\leq \sqrt{\frac{R}{2r}} \cdot h_a(h_b^2 + h_c^2) \leq \sqrt{\frac{R}{2r}} \cdot h_a(w_b^2 + w_c^2) \stackrel{\text{Lascu} + \text{AM-GM}}{\leq} \\ \sqrt{\frac{R}{2r}} \cdot h_a \cdot (s(s-b) + s(s-c)) &= \sqrt{\frac{R}{2r}} \cdot \frac{2rs}{a} \cdot sa \Rightarrow w_a(h_b^2 + h_c^2) \leq \sqrt{\frac{R}{2r}} \cdot 2s^2 r \\ \text{and analogs } \Rightarrow \left(\frac{R}{2r}\right)^3 \cdot \sum_{cyc} \frac{r_a^3 + r_a r_b r_c}{w_a(h_b^2 + h_c^2)} &\stackrel{①}{\geq} \frac{\left(\frac{R}{2r}\right)^3}{\sqrt{\frac{R}{2r}} \cdot 2s^2 r} \cdot \left(\sum_{cyc} r_a^3 + 3r_a r_b r_c \right) \\ \text{Again, } \sum_{cyc} \left(\sqrt{\frac{m_a}{h_a}} \cdot \frac{r_b r_c}{w_b w_c} \right) &\stackrel{\text{Panaitopol}}{\leq} \sqrt{\frac{R}{2r}} \cdot \sum_{cyc} \frac{s(s-a)}{h_b h_c} = \\ &\sqrt{\frac{R}{2r}} \cdot \frac{s}{4r^2 s^2} \cdot (s(s^2 + 4Rr + r^2) - 12Rrs) \\ \therefore \sum_{cyc} \left(\sqrt{\frac{m_a}{h_a}} \cdot \frac{r_b r_c}{w_b w_c} \right) &\stackrel{②}{\leq} \sqrt{\frac{R}{2r}} \cdot \frac{s^2 - 8Rr + r^2}{4r^2} \therefore ① \text{ and } ② \Rightarrow \text{it suffices to prove :} \\ \frac{\left(\frac{R}{2r}\right)^3}{2s^2 r} \cdot ((4R+r)^3 - (12R-3r)s^2) &\stackrel{?}{\geq} \frac{R}{2r} \cdot \frac{s^2 - 8Rr + r^2}{4r^2} \\ \Leftrightarrow R^2((4R+r)^3 - (12R-3r)s^2) &\stackrel{?}{\geq} 2s^2 r(s^2 - 8Rr + r^2) \\ \text{and } \therefore (12R-3r)s^2 &\stackrel{\text{Gerretsen}}{\leq} (12R-3r)(4R^2 + 4Rr + 3r^2) \text{ and } s^2(s^2 - 8Rr + r^2) \end{aligned}$$

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Gerretsen
 $\leq (4R^2 + 4Rr + 3r^2)(4R^2 - 4Rr + 4r^2) \therefore$ it suffices to prove :

$$R^2 \left((4R + r)^3 - (12R - 3r)(4R^2 - 4Rr + 4r^2) \right) \stackrel{?}{\geq}$$

$$2r(4R^2 - 4Rr + 4r^2)(4R^2 - 4Rr + 4r^2) \Leftrightarrow 8t^5 - 10t^4 - 6t^3 - 7t^2 - 4t - 12 \stackrel{?}{\geq} 0$$

$$\left(t = \frac{R}{r} \right) \Leftrightarrow (t - 2)(8t^4 + 6t^3 + 6t^2 + 5t + 6) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \text{ and so,}$$

$$\left(\frac{R}{2r} \right)^3 \cdot \sum_{\text{cyc}} \frac{r_a^3 + r_a r_b r_c}{w_a (h_b^2 + h_c^2)} \geq \sum_{\text{cyc}} \left(\sqrt{\frac{m_a}{h_a}} \cdot \frac{r_b r_c}{w_b w_c} \right) \forall \Delta ABC,$$

" = " iff ΔABC is equilateral (QED)