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In any ΔABC the following relationship holds :

$$\left(\frac{R}{2r}\right)^{\frac{3}{2}} \cdot \sum_{\text{cyc}} \frac{r_a^2 + r_b r_c}{w_a(h_b + h_c)} \geq \sum_{\text{cyc}} \sqrt{\frac{m_a}{h_a}}$$

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$$\begin{aligned} \sum_{\text{cyc}} \frac{r_a^2 + r_b r_c}{w_a(h_b + h_c)} &\stackrel{\text{Lascu} + \text{AM-GM}}{\geq} \sum_{\text{cyc}} \frac{r_a^2 + w_a^2}{\frac{2bc}{b+c} \cos \frac{A}{2} \cdot \frac{a(b+c)}{2R}} \stackrel{\text{Bergstrom}}{\geq} \sum_{\text{cyc}} \frac{r_a^2 + h_a^2}{4rs \cdot \cos \frac{A}{2}} \stackrel{\text{Doucet or Trucht and Gordon and Jensen}}{\geq} \\ &\frac{1}{4rs} \cdot \left(\frac{(4R+r)^2}{\sum_{\text{cyc}} \cos \frac{A}{2}} + \frac{(s^2 + 4Rr + r^2)^2}{4R^2 \cdot \sum_{\text{cyc}} \cos \frac{A}{2}} \right) \stackrel{\text{Gerretsen}}{\geq} \\ &\frac{1}{4rs} \cdot \left(\frac{(4R+r)(\sqrt{3} \cdot s)}{\frac{3\sqrt{3}}{2}} + \frac{(s^2 + 4Rr + r^2)(4\sqrt{3} \cdot sr)}{4R^2 \cdot \frac{3\sqrt{3}}{2}} \right) \stackrel{\text{Gerretsen}}{\geq} \\ &\frac{1}{6r} \cdot \left(\frac{R^2(4R+r) + r(20Rr - 4r^2)}{R^2} \right) \\ \Rightarrow \left(\frac{R}{2r}\right)^{\frac{3}{2}} \cdot \sum_{\text{cyc}} \frac{r_a^2 + r_b r_c}{w_a(h_b + h_c)} &\geq \left(\frac{R}{2r}\right)^{\frac{3}{2}} \cdot \frac{1}{6r} \cdot \left(\frac{R^2(4R+r) + r(20Rr - 4r^2)}{R^2} \right) \stackrel{?}{\geq} 3 \cdot \sqrt{\frac{R}{2r}} \\ &\Leftrightarrow (R^2 - 4r^2)(4R+r) \stackrel{?}{\geq} 0 \rightarrow \text{true via Euler} \\ \therefore \left(\frac{R}{2r}\right)^{\frac{3}{2}} \cdot \sum_{\text{cyc}} \frac{r_a^2 + r_b r_c}{w_a(h_b + h_c)} &\geq \sqrt{\frac{R}{2r}} + \sqrt{\frac{R}{2r}} + \sqrt{\frac{R}{2r}} \stackrel{\text{Panaitopol}}{\geq} \sum_{\text{cyc}} \sqrt{\frac{m_a}{h_a}} \text{ and so,} \\ \left(\frac{R}{2r}\right)^{\frac{3}{2}} \cdot \sum_{\text{cyc}} \frac{r_a^2 + r_b r_c}{w_a(h_b + h_c)} &\geq \sum_{\text{cyc}} \sqrt{\frac{m_a}{h_a}} \quad \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)} \end{aligned}$$