

ROMANIAN MATHEMATICAL MAGAZINE

In $\triangle ABC$ the following relationship holds:

$$\sum \left(\frac{m_a^4 + w_a^4 + h_a^4}{h_a^4} \cdot \frac{r_a^2}{(s-b)(s-c)} + \frac{(a+b)^2}{4ab} \cdot \frac{r_b + r_c}{r_a} \right) \geq 12 \cdot \frac{R}{r} + \sum \frac{r_b + r_c}{r_a} + 3 \geq 33$$

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Solution by Tapas Das-India

$$\begin{aligned} & \frac{(a+b)^2}{4ab} \stackrel{AM-GM}{\geq} 1 \\ & \frac{m_a^4 + w_a^4 + h_a^4}{h_a^4} \geq \frac{h_a^4 + h_a^4 + h_a^4}{h_a^4} = 3 \text{ (as } m_a \geq w_a \geq h_a) \\ & \sum \frac{r_a^2}{(s-b)(s-c)} = \sum \frac{F^2}{(s-a)^2(s-b)(s-c)} = \sum \frac{s}{s-a} = \frac{s(4R+r)}{sr} = 4\left(\frac{R}{r}\right) + 1 \\ & \sum \left(\frac{m_a^4 + w_a^4 + h_a^4}{h_a^4} \cdot \frac{r_a^2}{(s-b)(s-c)} + \frac{(a+b)^2}{4ab} \cdot \frac{r_b + r_c}{r_a} \right) = \\ & = \sum \left(\frac{m_a^4 + w_a^4 + h_a^4}{h_a^4} \cdot \frac{r_a^2}{(s-b)(s-c)} \right) + \sum \left(\frac{(a+b)^2}{4ab} \cdot \frac{r_b + r_c}{r_a} \right) \geq \\ & \geq \sum \left(3 \frac{r_a^2}{(s-b)(s-c)} \right) + \sum \left(1 \times \frac{r_b + r_c}{r_a} \right) = 3 \left(\frac{4R}{r} + 1 \right) + \sum \frac{r_b + r_c}{r_a} = \\ & = 12 \cdot \frac{R}{r} + \sum \frac{r_b + r_c}{r_a} + 3 \stackrel{AM-GM}{\geq} 12 \cdot \frac{R}{r} + \sum \frac{2\sqrt{r_b r_c}}{r_a} + 3 \stackrel{Euler}{\geq} \\ & \geq 12 \times 2 + 2 \times 3 + 3 = 33 \end{aligned}$$

Equality holds for an equilateral triangle.