

ROMANIAN MATHEMATICAL MAGAZINE

In $\triangle ABC$ the following relationship holds:

$$\sum \left(\frac{m_a^4 + w_a^4 + g_a^4}{3h_a^4} \cdot \left(\frac{r_a}{r} \right)^2 \right) + \frac{ab + bc + ca - 4Rr - r^2}{27r^2} \geq 4 \left(2 \left(\frac{R}{r} \right)^2 - 1 \right) \geq 4 \left(\frac{4R}{r} - 1 \right) \geq 28$$

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$$m_a \geq h_a, g_a \geq h_a, w_a \geq h_a$$

$$\frac{m_a^4 + w_a^4 + g_a^4}{3h_a^4} = \frac{1}{3} \left(\left(\frac{m_a}{h_a} \right)^4 + \left(\frac{w_a}{h_a} \right)^4 + \left(\frac{g_a}{h_a} \right)^4 \right) \geq \frac{1}{3} (1 + 1 + 1) = 1$$

$$\begin{aligned} \sum \left(\frac{m_a^4 + w_a^4 + g_a^4}{3h_a^4} \cdot \left(\frac{r_a}{r} \right)^2 \right) &\geq \sum \left(\left(\frac{r_a}{r} \right)^2 \right) = \frac{1}{r^2} ((4R + r)^2 - 2s^2) = \\ &= \frac{27(4R + r)^2 - 54s^2}{27r^2} \end{aligned}$$

$$\frac{ab + bc + ca - 4Rr - r^2}{27r^2} = \frac{s^2 + r^2 + 4Rr - 4Rr - r^2}{27r^2} = \frac{s^2}{27r^2}$$

$$\begin{aligned} \sum \left(\frac{m_a^4 + w_a^4 + g_a^4}{3h_a^4} \cdot \left(\frac{r_a}{r} \right)^2 \right) + \frac{ab + bc + ca - 4Rr - r^2}{27r^2} &\geq \frac{27(4R + r)^2 - 54s^2}{27r^2} + \frac{s^2}{27r^2} \\ &= \frac{27((4R + r)^2 - 2s^2)}{27r^2} + \frac{s^2}{27r^2} \stackrel{\text{Gerretsn Mitrinovic}}{\geq} \frac{(4R + r)^2 - 2(4R^2 + 4Rr + 3r^2)}{r^2} + \frac{27r^2}{27r^2} = \\ &= \frac{8R^2 - 5r^2}{r^2} + 1 = 8 \left(\frac{R}{r} \right)^2 - 4 = 4 \left(2 \left(\frac{R}{r} \right)^2 - 1 \right) = \\ &= 4 \left(\frac{2R}{r} \cdot \frac{R}{r} - 1 \right) \stackrel{\text{Euler}}{\geq} 4 \left(2 \times 2 \times \frac{R}{r} - 1 \right) = 4 \left(4 \cdot \frac{R}{r} - 1 \right) \stackrel{\text{Euler}}{\geq} 4(4 \times 2 - 1) = 28 \end{aligned}$$

Equality holds for an equilateral triangle.