

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$\frac{r_a^4 h_b^2 h_c^2}{h_a^4 (r_b^2 h_c^2 + r_c^2 h_b^2)} + \frac{r_b^4 h_c^2 h_a^2}{h_b^4 (r_c^2 h_a^2 + r_a^2 h_c^2)} + \frac{r_c^4 h_a^2 h_b^2}{h_c^4 (r_a^2 h_b^2 + r_b^2 h_a^2)} \geq \frac{(2R - r)^2}{6r^2}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum_{\text{cyc}} \frac{r_a^4 h_b^2 h_c^2}{h_a^4 (r_b^2 h_c^2 + r_c^2 h_b^2)} &= \sum_{\text{cyc}} \frac{\left(\frac{r_a^2}{h_a^2}\right)^2}{\frac{r_b^2}{h_b^2} + \frac{r_c^2}{h_c^2}} \stackrel{\text{Bergstrom}}{\geq} \frac{\left(\sum_{\text{cyc}} \frac{r_a^2}{h_a^2}\right)^2}{2 \sum_{\text{cyc}} \frac{r_a^2}{h_a^2}} = \frac{1}{2} \cdot \sum_{\text{cyc}} \frac{r_a^2}{h_a^2} \\ &\geq \frac{1}{6} \cdot \left(\sum_{\text{cyc}} \frac{r_a}{h_a}\right)^2 = \frac{1}{6} \cdot \left(\sum_{\text{cyc}} \frac{s \tan \frac{A}{2} \cdot 4R \cdot \cos \frac{A}{2} \sin \frac{A}{2}}{2rs}\right)^2 = \frac{1}{6} \cdot \left(\frac{2R}{r} \cdot \sum_{\text{cyc}} \sin^2 \frac{A}{2}\right)^2 \\ &= \frac{1}{6} \cdot \left(\frac{2R}{r} \cdot \frac{2R - r}{2R}\right)^2 = \frac{(2R - r)^2}{6r^2} \text{ and so,} \\ \frac{r_a^4 h_b^2 h_c^2}{h_a^4 (r_b^2 h_c^2 + r_c^2 h_b^2)} + \frac{r_b^4 h_c^2 h_a^2}{h_b^4 (r_c^2 h_a^2 + r_a^2 h_c^2)} + \frac{r_c^4 h_a^2 h_b^2}{h_c^4 (r_a^2 h_b^2 + r_b^2 h_a^2)} &\geq \frac{(2R - r)^2}{6r^2} \\ \forall \Delta ABC, '' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)} \end{aligned}$$