

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$8 \prod_{\text{cyc}} (m_a + m_b) \leq \prod_{\text{cyc}} (r_a + r_b + 2r_c)$$

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$$\begin{aligned} 64 \prod_{\text{cyc}} (m_b + m_c)^2 &= 64 \prod_{\text{cyc}} (m_b^2 + m_c^2 + 2m_b m_c) \leq \\ \prod_{\text{cyc}} (2c^2 + 2a^2 - b^2 + 2a^2 + 2b^2 - c^2 + 4a^2 + 2bc) &= \prod_{\text{cyc}} \left(\sum_{\text{cyc}} a^2 + 7a^2 + 2bc \right) \\ &= \left(\sum_{\text{cyc}} a^2 \right)^3 + \left(\sum_{\text{cyc}} a^2 \right) \left(49 \sum_{\text{cyc}} a^2 b^2 + 14 \left(\sum_{\text{cyc}} ab \right) \left(\sum_{\text{cyc}} a^2 \right) - 10abc \left(\sum_{\text{cyc}} a \right) \right) \\ &\quad + \left(\sum_{\text{cyc}} a^2 \right)^2 \left(7 \sum_{\text{cyc}} a^2 + 2 \sum_{\text{cyc}} ab \right) + 28abc \sum_{\text{cyc}} a^3 + 98 \sum_{\text{cyc}} a^3 b^3 + 351a^2 b^2 c^2 \\ &= 8(s^2 - 4Rr - r^2)^3 + \\ 2(s^2 - 4Rr - r^2) &\left(49((s^2 + 4Rr + r^2)^2 - 16Rrs^2) + 28(s^4 - (4Rr + r^2)^2) - \right) + \\ &\quad 80Rrs^2 \\ &\quad 4(s^2 - 4Rr - r^2)^2 \left(14(s^2 - 4Rr - r^2) + 2(s^2 + 4Rr + r^2) \right) + \\ 28(4Rrs)(2s)(s^2 - 6Rr - 3r^2) &+ 98 \left((s^2 + 4Rr + r^2)^3 - 24Rrs^2(s^2 + 2Rr + r^2) \right) \\ &\quad + 351(16R^2 r^2 s^2) \stackrel{?}{\leq} \prod_{\text{cyc}} (r_a + r_b + 2r_c)^2 = \left(\prod_{\text{cyc}} (4R + r + r_a) \right)^2 \\ &= 4((4R + r)^3 + (2R + r)s^2)^2 \Leftrightarrow -81s^6 + (4R^2 + 832Rr - 33r^2)s^4 + \\ &\quad (256R^4 + 320R^3r - 2772R^2r^2 - 296Rr^3 - 79r^4)s^2 + (4R + r)^6 \stackrel{?}{\geq} 0 \end{aligned}$$

Now, Rouché $\Rightarrow s^2 - (m - n) \geq 0$ and $s^2 - (m + n) \leq 0$, where $m = 2R^2 + 10Rr - r^2$ & $n = 2(R - 2r)\sqrt{R^2 - 2Rr} \therefore (s^2 - (m + n))(s^2 - (m - n)) \leq 0$
 $\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0 \Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3 \stackrel{(*)}{\leq} 0$
 $\therefore$ in order to prove ①, it suffices to prove :

$$\begin{aligned} \text{LHS of ①} &\stackrel{?}{\geq} -81s^2(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3) \\ &\Leftrightarrow -(320R^2 + 788Rr - 129r^2)s^4 + \end{aligned}$$

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$(256R^4 + 5504R^3r + 1116R^2r^2 + 676Rr^3 + 2r^4)s^2 + (4R + r)^6 \stackrel{?}{\geq} 0$ and again,

since : $P = -(320R^2 + 788Rr - 129r^2)(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3)$
 via (*) $\geq 0 \therefore$ in order to prove ②, it suffices to prove : LHS of ② $\stackrel{?}{\geq} P \Leftrightarrow$

$$256R^6 + 1664R^5r + 4352R^4r^2 + 2168R^3r^3 + 239R^2r^4 - 46Rr^5 - 8r^6 \stackrel{?}{\geq}$$

$$(64R^4 + 253R^3r + 843R^2r^2 - 302Rr^3 + 16r^4)s^2$$

Gerretsen
 Finally, RHS of ③ $\leq$

$$(64R^4 + 253R^3r + 843R^2r^2 - 302Rr^3 + 16r^4)(4R^2 + 4Rr + 3r^2) \stackrel{?}{\leq} \text{LHS of ③}$$

$$\Leftrightarrow 396t^5 - 224t^4 - 755t^3 - 1146t^2 + 796t - 56 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)(396t^4 + 568t^3 + 381t^2 - 384t + 28) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2$$

$$\Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true } \therefore 8 \prod_{\text{cyc}} (m_a + m_b) \leq \prod_{\text{cyc}} (r_a + r_b + 2r_c) \quad \forall \Delta ABC,$$

" = " iff ΔABC is equilateral (QED)