

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$2 \sum_{cyc} (w_a bc) \leq \prod_{cyc} (r_a + r_b) \leq \frac{8}{3} \sum_{cyc} (r_a m_b m_c)$$

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$$\begin{aligned} 2m_b m_c &\stackrel{?}{\geq} \frac{9}{4} a^2 + h_a^2 - m_b^2 - m_c^2 = \frac{9a^2 - 4a^2 - b^2 - c^2}{4} + \frac{4F^2}{a^2} \\ &= \frac{a^2(5a^2 - b^2 - c^2) + 2 \sum_{cyc} a^2 b^2 - \sum_{cyc} a^4}{4a^2} \\ &= \frac{a^2(4a^2 + b^2 + c^2) - (b^2 - c^2)^2}{4a^2} \text{ and it's trivially true if :} \\ &\quad a^2(4a^2 + b^2 + c^2) - (b^2 - c^2)^2 \leq 0 \text{ and when :} \\ &\quad a^2(4a^2 + b^2 + c^2) - (b^2 - c^2)^2 > 0, \text{ then we are to prove :} \\ 4a^4(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2) &\stackrel{?}{\geq} (a^2(4a^2 + b^2 + c^2) - (b^2 - c^2)^2)^2 \\ \Leftrightarrow -a^4(b^2 - c^2)^2 + 2a^2(b^2 + c^2)(b^2 - c^2)^2 - (b^2 - c^2)^4 &\stackrel{?}{\geq} 0 \\ \Leftrightarrow (b^2 - c^2)^2(-a^4 + 2a^2(b^2 + c^2) - b^4 - c^4 + 2b^2c^2) &\stackrel{?}{\geq} 0 \\ \Leftrightarrow (b^2 - c^2)^2 \cdot 16F^2 &\stackrel{?}{\geq} 0 \rightarrow \text{true} \therefore 2m_b m_c \geq \frac{9}{4} a^2 + h_a^2 - m_b^2 - m_c^2 \\ &= \frac{6a^2 - \sum_{cyc} a^2}{4} + h_a^2 \text{ and analogs} \Rightarrow \frac{8}{3} \sum_{cyc} (r_a m_b m_c) \geq \\ &\quad \frac{4}{3} \sum_{cyc} \left(\left(\frac{6a^2 - \sum_{cyc} a^2}{4} + h_a^2 \right) r_a \right) = \\ &\quad \frac{3rs}{2} \cdot \frac{4}{3} \sum_{cyc} \frac{(a^2 - s^2) + s^2}{s - a} - \frac{2(s^2 - 4Rr - r^2)(4R + r)}{3} + \\ &\quad \frac{rs}{3R^2 r^2 s} \sum_{cyc} (b^2 c^2 (-s^2 + sa + bc)) = 2rs(-5s) + \frac{2s(4Rr + r^2)}{r^2} - \\ &\quad \frac{2(s^2 - 4Rr - r^2)(4R + r)}{3} + \\ &\quad -s^2((s^2 + 4Rr + r^2)^2 - 16Rrs^2) + 4Rrs^2(s^2 + 4Rr + r^2) + \\ &\quad \frac{(s^2 + 4Rr + r^2)^3 - 24Rrs^2(s^2 + 2Rr + r^2)}{3R^2 r} \\ &= \frac{24R^2(R - r)s^2 - 2R^2(4R + r)(s^2 - 4Rr - r^2) + r(s^4 - (4Rr - 2r^2)s^2 + r(4R + r)^3)}{3R^2} \end{aligned}$$

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$$\geq \prod_{cyc} (r_a + r_b) = 64R^3 \cdot \frac{s^2}{16R^2} \Leftrightarrow rs^4 + (4R^3 - 26R^2r - 4Rr^2 + 2r^3)s^2 +$$

$$r(32R^4 + 80R^3r + 50R^2r^2 + 12Rr^3 + r^4) \stackrel{?}{\geq} 0 \text{ and } \therefore (s^2 - 16Rr + 5r^2)^2$$

Gerretsen
 $\geq 0 \therefore$ in order to prove ①, it suffices to prove :

$$\text{LHS of ①} \stackrel{?}{\geq} r(s^2 - 16Rr + 5r^2)^2 \Leftrightarrow (2R^3 - 13R^2r + 14Rr^2 - 4r^3)s^2 +$$

$$r(16R^4 + 40R^3r - 103R^2r^2 + 86Rr^3 - 12r^4) \stackrel{?}{\geq} 0$$

$$\text{Now, } 16R^4 + 40R^3r - 103R^2r^2 + 86Rr^3 - 12r^4 =$$

$(R - 2r)(16R^3 + 72R^2r + 41Rr^2 + 168r^3) + 324r^4 \stackrel{\text{Euler}}{\geq} 324r^4 > 0$ and so,
 if : $2R^3 - 13R^2r + 14Rr^2 - 4r^3 \geq 0$, then : ② is trivially true and when :

$$2R^3 - 13R^2r + 14Rr^2 - 4r^3 < 0, \text{ then : LHS of ②} \stackrel{\text{Gerretsen}}{\geq} (2R^3 - 13R^2r + 14Rr^2 - 4r^3)(4R^2 + 4Rr + 3r^2) +$$

$$r(16R^4 + 40R^3r - 103R^2r^2 + 86Rr^3 - 12r^4) \stackrel{?}{\geq} 0$$

$$\Leftrightarrow 4t^5 - 14t^4 + 25t^3 - 51t^2 + 56t - 12 \geq 0 \left(t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)(t^4 + 3t^3(t - 2) + 13t(t - 2) + t + 6) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2$$

$\Rightarrow$ ② $\Rightarrow$ ① is true and again,

$$2 \sum_{cyc} (w_a bc) \stackrel{\text{Lascu} + \text{AM-GM}}{\leq} 2 \sum_{cyc} (\sqrt{s(s-a)} \cdot bc) = 2 \sum_{cyc} (\sqrt{s(s-a)} bc \cdot \sqrt{bc})$$

$$\stackrel{\text{CBS}}{\leq} 2\sqrt{s^2(s^2 + 4Rr + r^2) - 12Rrs^2} \cdot \sqrt{s^2 + 4Rr + r^2} \stackrel{?}{\leq} \prod_{cyc} (r_a + r_b) = 64R^3 \cdot \frac{s^2}{16R^2}$$

$$\Leftrightarrow -s^4 + (4R^2 + 4Rr - 2r^2)s^2 + r^2(32R^2 + 4Rr - r^2) \stackrel{?}{\geq} 0$$

$$\text{Now, LHS of ③} \stackrel{\text{Gerretsen}}{\geq} - (4R^2 + 4Rr + 3r^2)s^2 + (4R^2 + 4Rr - 2r^2)s^2 +$$

$$r^2(32R^2 + 4Rr - r^2) \stackrel{?}{\geq} 0 \Leftrightarrow 32R^2 + 4Rr - r^2 \stackrel{?}{\geq} 5s^2 \text{ \& finally, } 5s^2 \stackrel{\text{Gerretsen}}{\leq}$$

$$5(4R^2 + 4Rr + 3r^2) \stackrel{?}{\leq} 32R^2 + 4Rr - r^2 \Leftrightarrow 4(R - 2r)(3R + 2r) \stackrel{?}{\geq} 0 \rightarrow \text{true}$$

$$\Rightarrow \text{④} \Rightarrow \text{③ is true} \therefore 2 \sum_{cyc} (w_a bc) \leq \prod_{cyc} (r_a + r_b) \leq \frac{8}{3} \sum_{cyc} (r_a m_b m_c),$$

" = " iff ΔABC is equilateral (QED)