

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC following relationship holds:

$$\max \left\{ \frac{1}{\sqrt{r_a} + \sqrt{r_b}} + \frac{1}{\sqrt{r_a} + \sqrt{r_c}}, \frac{1}{2\sqrt{r_a}} + \frac{1}{\sqrt{r_b} + \sqrt{r_c}} \right\} \leq \frac{2}{\sqrt{h_b} + \sqrt{h_c}}$$

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$$\text{Let } x = \frac{1}{\sqrt{r_a}}, y = \frac{1}{\sqrt{r_b}}, z = \frac{1}{\sqrt{r_c}}$$

$$\frac{1}{r_a} + \frac{1}{r_c} = \frac{1}{F}(s-a+s-c) = \frac{b}{F} = \frac{2}{h_b} \text{ or, } \frac{1}{h_b} = \frac{1}{2} \left(\frac{1}{r_a} + \frac{1}{r_c} \right) = \frac{x^2 + z^2}{2}$$

$$\sqrt{h_b} = \frac{\sqrt{2}}{\sqrt{x^2 + z^2}}. \text{ Similarly: } \frac{1}{r_a} + \frac{1}{r_b} = \frac{2}{h_c} \text{ or } \sqrt{h_c} = \frac{\sqrt{2}}{\sqrt{x^2 + y^2}}$$

$$\text{Let } T_1 = \frac{1}{\sqrt{r_a} + \sqrt{r_c}} + \frac{1}{\sqrt{r_a} + \sqrt{r_b}} = \frac{xy}{x+y} + \frac{xz}{x+z} =$$

$$= \left(x - \frac{x^2}{x+y} \right) + \left(x - \frac{x^2}{x+z} \right) = 2x - x^2 \left(\frac{1}{x+y} + \frac{1}{x+z} \right) \&$$

$$T_2 = \frac{1}{2\sqrt{r_a}} + \frac{1}{\sqrt{r_c} + \sqrt{r_b}} = \frac{x}{2} + \frac{yz}{y+z}$$

$$\sqrt{x^2 + y^2} \geq \frac{x+y}{\sqrt{2}} \text{ or, } \frac{1}{\sqrt{x^2 + y^2}} \leq \frac{\sqrt{2}}{x+y} \& \frac{1}{\sqrt{x^2 + z^2}} \leq \frac{\sqrt{2}}{x+z}$$

$$R.H.S = \frac{2}{\sqrt{h_b} + \sqrt{h_c}} = \frac{\sqrt{2}}{\frac{1}{\sqrt{x^2 + z^2}} + \frac{1}{\sqrt{x^2 + y^2}}} \geq \frac{\sqrt{2}}{\frac{\sqrt{2}}{x+y} + \frac{\sqrt{2}}{x+z}} = \frac{(x+y)(x+z)}{2x+y+z}$$

$$\text{We need to show: } T_1 \leq \frac{2}{\sqrt{h_b} + \sqrt{h_c}}$$

$$2x - x^2 \left(\frac{1}{x+y} + \frac{1}{x+z} \right) \leq \frac{(x+y)(x+z)}{2x+y+z}$$

$$\text{or, } 2x - x^2 u \stackrel{u = \left(\frac{1}{x+y} + \frac{1}{x+z} \right)}{\leq} \frac{1}{u} \text{ or, } 2xu - x^2 u^2 \leq 1 \text{ or, } (xu - 1)^2 \geq 0 \text{ True}$$

$$\text{We need to show: } T_2 \leq \frac{2}{\sqrt{h_b} + \sqrt{h_c}} \text{ or, } \frac{x}{2} + \frac{yz}{y+z} \leq \frac{(x+y)(x+z)}{2x+y+z}$$

$$\text{or, } \frac{yz}{y+z} \leq \frac{x(y+z) + 2yz}{2(2x+y+z)} \text{ or, } 2yz(2x+y+z) \leq (y+z)(x(y+z) + 2yz)$$

$$4yz \leq (y+z)^2 \text{ or } (y-z)^2 \geq 0 \text{ true}$$

Equality holds for an equilateral triangle.