

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC with $a \geq b \geq c$, $d \leq e \leq f$ holds:

$$d(r_b + r_c - 2m_a) + e(r_c + r_a - 2m_b) + f(r_a + r_b - 2m_c) \geq 0$$

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$$r_a + r_c = \frac{F}{s-a} + \frac{F}{s-b} = F \frac{2s-a-c}{(s-a)(s-c)} = b \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = bcot\left(\frac{B}{2}\right)$$

We will show:

$$bcot\left(\frac{B}{2}\right) \geq 2m_b \text{ or } b^2 \cot^2 \frac{B}{2} \geq 2a^2 + 2c^2 - b^2$$

$$b^2 \left(1 + \cot^2 \frac{B}{2}\right) \geq 2a^2 + 2c^2 \text{ or } b^2 \geq 2 \sin^2 \frac{B}{2} (a^2 + c^2)$$

$$b^2 \geq 2(1 - \cos B)(a^2 + c^2)$$

$$\text{or, } b^2 \geq \frac{b^2 - (a-c)^2}{2ac} (a^2 + c^2) \text{ or, } 2acb^2 \geq b^2(a^2 + c^2) - (a-c)^2(a^2 + c^2)$$

$$b^2(a-c)^2 - (a-c)^2(a^2 + c^2) \geq 0$$

$$(a-c)^2(a^2 + c^2 - b^2) \geq 0 \text{ True as } (a \geq b \geq c)$$

so, $r_a + r_c \geq 2m_b$ & similarly, $r_a + r_b \geq 2m_c$ as $a \geq b \geq c$

let, $x = (r_b + r_c - 2m_a)$, $y = (r_c + r_a - 2m_b) > 0$, $z = (r_a + r_b - 2m_c) > 0$ then

$$x + y + z = 2(r_a + r_b + r_c) - 2(m_a + m_b + m_c) \stackrel{\text{Leuanberger}}{\geq} \geq 2(4R + r) - 2(4R + r) = 0$$

$$d(r_b + r_c - 2m_a) + e(r_c + r_a - 2m_b) + f(r_a + r_b - 2m_c) = dx + ey + fz =$$

$$= d(x + y + z) + (e-d)y + (f-e)z \geq 0 \text{ true as } d \geq 0, e-d \geq 0, f-e \geq 0 \text{ as } d \leq e \leq f$$

Equality holds for $d=e=f=0$.