

ROMANIAN MATHEMATICAL MAGAZINE

In $\triangle ABC$ the following relationship holds:

$$\frac{1}{r_b + r_c} + \frac{1}{r_a} \geq \frac{\sqrt{3}}{a}$$

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$$\begin{aligned} \frac{1}{r_b + r_c} + \frac{1}{r_a} &= \frac{(s-b)(s-c)}{aF} + \frac{s-a}{F} = \frac{s^2 - 2s^2 + 2as + bc - a^2}{aF} = \\ &= \frac{2as + bc - s^2 - a^2}{aF} = \frac{(a+b+c)a + bc - \frac{(a+b+c)^2}{4} - a^2}{aF} = \\ &= \frac{4(ab + bc + ca) - (a+b+c)^2}{4aF} = \frac{2\sum ab - \sum a^2}{4aF} \end{aligned}$$

We need to show:

$$\begin{aligned} \frac{1}{r_b + r_c} + \frac{1}{r_a} &\geq \frac{\sqrt{3}}{a} \text{ or } \frac{2\sum ab - \sum a^2}{4aF} \geq \frac{\sqrt{3}}{a} \\ 2\sum ab - \sum a^2 &\geq 4\sqrt{3}F \text{ true} \end{aligned}$$

Hadwiger – Finsler inequality:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}F + (a-b)^2 + (b-c)^2 + (c-a)^2$$

$$\sum a^2 \geq 4\sqrt{3}F + 2\sum a^2 - 2\sum ab \text{ or } 2\sum ab - \sum a^2 \geq 4\sqrt{3}F$$

Equality holds for an equilateral triangle.