

ROMANIAN MATHEMATICAL MAGAZINE

In $\triangle ABC$ the following relationship holds:

$$4w_a + 3\sqrt{3}a \leq r_a + 10R + 7r$$

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$$\text{Let } x = \frac{A}{2}, d = \frac{B-C}{2} \& \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2} - x \& B, C > 0 \text{ then } |d| \leq \frac{\pi}{2} - x$$

$$\text{and } \cos d = \cos \frac{B-C}{2} \geq \cos \left(\frac{\pi}{2} - x \right) = \sin x, \text{ let, } s = \sin x, c = \cos d \text{ then}$$

$$0 < s \leq c \leq 1$$

$$a = 2R \sin A = 4R \sin x \cdot \cos x$$

$$r = 4R \sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right) \sin \left(\frac{C}{2} \right) = 2Rs(c-s) \left(\text{as } \sin \left(\frac{B}{2} \right) \sin \left(\frac{C}{2} \right) = \frac{\cos d - \sin x}{2} \right)$$

$$r_a = 4R \sin \left(\frac{A}{2} \right) \cos \left(\frac{B}{2} \right) \cos \left(\frac{C}{2} \right) = 2Rs(c+s) \left(\text{as } \cos \left(\frac{B}{2} \right) \cos \left(\frac{C}{2} \right) = \frac{(c+s)}{2} \right)$$

$$\sin B + \sin C \stackrel{A+B+C=\pi}{=} 2 \cos \left(\frac{A}{2} \right) \cdot \cos \left(\frac{B-C}{2} \right) = 2c \cdot \cos x$$

$$bc = 4R^2 \sin B \cdot \sin C = 2R^2 (\cos(B-C) - \cos(B+C))$$

$$= 2R^2 \left(2 \cos^2 \frac{B-C}{2} - 1 + \cos A \right)$$

$$= 2R^2 \left(2 \cos^2 \frac{B-C}{2} - 1 + 1 - 2 \sin^2 \frac{A}{2} \right) = 4R^2 (c^2 - s^2)$$

$$\text{Using above result, } w_a = \frac{2bc \cos \left(\frac{A}{2} \right)}{b+c} = 2R \left(c - \frac{s^2}{c} \right) \& 3\sqrt{3}a = 12\sqrt{3}R \cos x$$

$$r_a + 10R + 7r = 2Rs(c+s) + 10R + 14Rs(c-s) = R(10 + 16sc - 12s^2)$$

$$\text{We need to show: } 4w_a + 3\sqrt{3}a \leq r_a + 10R + 7r$$

$$8R \left(c - \frac{s^2}{c} \right) + 12\sqrt{3}R \cos x \leq R(10 + 16sc - 12s^2)$$

$$10 - 8c + 16sc - 12s^2 + \frac{8s^2}{c} - 12\sqrt{3} \cos x \geq 0$$

$$10 - 8c + 16sc - 12s^2 + \frac{8s^2}{c} - 12\sqrt{3} s \sqrt{1 - \sin^2 x} \geq 0$$

$$10 - 8c + 16sc - 12s^2 + \frac{8s^2}{c} - 12\sqrt{3} s \sqrt{1 - s^2} \geq 0$$

$$\text{Let } F(s, c) = 10 - 8c + 16sc - 12s^2 + \frac{8s^2}{c} - 12\sqrt{3} s \sqrt{1 - s^2}$$

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We proved that $0 < s \leq c \leq 1$, so now we calculate

$$F(s, c) - F(s, 1) = 8(1 - c) - 16s(1 - c) + \frac{8s^2(1 - c)}{c} = 8(1 - c) \left(1 - 2s + \frac{s^2}{c} \right)$$

$$= 8(1 - c) \cdot \frac{(c - s)^2 + c(1 - c)}{c} \geq 0 \text{ as } 0 < s \leq c \leq 1$$

so, $F(s, c) - F(s, 1) \geq 0$ or, $F(s, c) \geq F(s, 1)$, now we will show $F(s, 1) \geq 0$,

$$F(s, 1) \geq 0 \text{ or, } 2 + 16s - 4s^2 - 12\sqrt{3}s\sqrt{1 - s^2} \geq 0$$

$$2 + 16s - 4s^2 \geq 12\sqrt{3}s\sqrt{1 - s^2} \text{ or, } (1 + 8s - 2s^2)^2 \geq 108s^2(1 - s^2)$$

$$4s^4 - 32s^3 + 60s^2 + 16s + 1 \geq 0 \text{ or, } (2s - 1)^2(28s^2 + 20s + 1) \geq 0 \text{ true}$$

$$\text{so, } F(s, c) \geq F(s, 1) \geq 0 \text{ or,}$$

$$10 - 8c + 16sc - 12s^2 + \frac{8s^2}{c} - 12\sqrt{3}s\sqrt{1 - s^2} \geq 0$$

$$\text{or, } 4w_a + 3\sqrt{3}a \leq r_a + 10R + 7r$$

Equality holds for an equilateral triangle.