

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds

$$2(m_b^2 + m_c^2) \leq 2r_a^2 + r_b^2 + r_c^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Case 1 $\hat{A}$ is non-acute and we assign : $S = \sin \frac{A}{2}$, $C = \cos \frac{B-C}{2}$ and

$$2(m_b^2 + m_c^2) \stackrel{?}{\leq} 2r_a^2 + r_b^2 + r_c^2 \Leftrightarrow 2 \sum_{cyc} m_a^2 - 2m_a^2 \stackrel{?}{\leq} 2 \sum_{cyc} r_a^2 - (r_b^2 + r_c^2)$$

$$\Leftrightarrow 2 \left(\sum_{cyc} r_a^2 - \sum_{cyc} m_a^2 \right) + 2m_a^2 \stackrel{?}{\geq} r_b^2 + r_c^2$$

Gerretsen
and
Leibnitz

$$\text{Now, } \sum_{cyc} r_a^2 - \sum_{cyc} m_a^2 \stackrel{?}{\geq} (4R+r)^2 - 2(4R^2 + 4Rr + 3r^2) - \frac{3}{4} \cdot 9R^2$$

$$= \frac{5(R^2 - 4r^2)}{4} \stackrel{\text{Euler}}{\geq} 0 \therefore \text{in order to prove } (*), \text{ it suffices to prove :}$$

$$2s(s-a) + \frac{(b-c)^2}{2} \stackrel{?}{\geq} (r_b + r_c)^2 - 2s(s-a)$$

$$\Leftrightarrow 4 \cdot 16R^2(1-S^2) \left(\frac{C+S}{2} \right) \left(\frac{C-S}{2} \right) + 8R^2S^2(1-C^2) \stackrel{?}{\geq} 16R^2(1-S^2)^2$$

$$\Leftrightarrow (1-C^2)(3S^2-2) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because 1 \geq C = \cos \frac{B-C}{2} \text{ and } \because \frac{A}{2} \geq \frac{\pi}{4}$$

$$\Rightarrow S \geq \frac{1}{\sqrt{2}} > \frac{\sqrt{2}}{\sqrt{3}} \Rightarrow 3S^2 - 2 > 0 \therefore 2(m_b^2 + m_c^2) < 2r_a^2 + r_b^2 + r_c^2$$

and the inequality is strict because $a = b = c$ cannot occur when $\hat{A}$ is non-acute

Case 2 $\hat{A}$ is acute and $2(m_b^2 + m_c^2) \stackrel{?}{\leq} 2r_a^2 + r_b^2 + r_c^2$

$$\Leftrightarrow 4a^2 + (b+c)^2 - 2bc \stackrel{?}{\leq} 4r_a^2 + 2(r_b + r_c)^2 - (b+c)^2 + a^2$$

$$\Leftrightarrow 3 \cdot 16R^2 \cdot S^2(1-S^2) + 2 \cdot 16R^2 \cdot C^2(1-S^2) - 8R^2(C^2 - S^2) \stackrel{?}{\leq} 16R^2S^2(C+S)^2 +$$

$$32R^2(1-S^2)^2 \Leftrightarrow 3(1-2S^2)C^2 - 4S^3 \cdot C + 15S^2 - 12S^4 - 4 \stackrel{?}{\geq} 0$$

(**)

Now, LHS of (**) is a quadratic polynomial in "C" with discriminant $\delta = 16S^6 - 12(1-2S^2)(15S^2 - 12S^4 - 4) = 2(-136x^3 + 252x^2 - 138x + 24)$

$$\left(x = S^2 < \frac{1}{2}; \text{ since } \frac{A}{2} < \frac{\pi}{4} \Rightarrow S < \frac{1}{\sqrt{2}} \right)$$

$$= 2(-17(1-t)^3 + 63(1-t)^2 - 69(1-t) + 24)$$

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$(\because 0 < 2x < 1, \text{ hence we can assign } 2x = 1 - t \text{ } (0 < t < 1))$
 $= 17t^3 + 3t^2 + (3t - 1)^2 > 0$ and $\because 1 - 2S^2 > 0 \therefore$ in order to prove (*),
 it suffices to prove : $6(1 - 2S^2)C \overset{?}{\underset{①}{\geq}} 4S^3 + \sqrt{\delta}$ AND $6(1 - 2S^2)C \overset{?}{\underset{②}{\geq}} 4S^3 - \sqrt{\delta}$
 Since $1 - 2S^2 > 0$ and $C \leq 1 \therefore 6(1 - 2S^2)C - 4S^3 \leq 6 - 12S^2 - 4S^3 = p$ (say)
 and if : $p \leq 0$, then $\because 0 < \sqrt{\delta} \therefore$ ① is trivially true and when : $p > 0$, then : ①
 $\Leftrightarrow 16S^6 - 12(1 - 2S^2)(15S^2 - 12S^4 - 4) \overset{?}{\geq} (6 - 12S^2 - 4S^3)^2$
 $\Leftrightarrow 12(1 - 2S^2)(S + 1)(3S + 1)(2S - 1)^2 \overset{?}{\geq} 0 \rightarrow \text{true} \because 0 < S < \frac{1}{\sqrt{2}} \Rightarrow$ ① is true
 Again, since $1 - 2S^2 > 0$ and $C > S \therefore 6(1 - 2S^2)C > 6(1 - 2S^2)S \overset{?}{\geq} 4S^3 - \sqrt{\delta}$
 $\Leftrightarrow \sqrt{\delta} \overset{?}{\geq} 2S(8S^2 - 3)$ and it's trivially true when : $S \leq \sqrt{\frac{3}{8}}$ ($\because \sqrt{\delta} > 0$)
 and when : $S > \sqrt{\frac{3}{8}}$, then : $\sqrt{\delta} \overset{?}{\geq} 2S(8S^2 - 3)$
 $\Leftrightarrow 16S^6 - 12(1 - 2S^2)(15S^2 - 12S^4 - 4) \overset{?}{\geq} 4S^2(8S^2 - 3)^2$
 $\Leftrightarrow \frac{-88x^3 + 116x^2 - 52x + 8}{4} \overset{?}{\geq} 0$
 $\Leftrightarrow -11(1 - t)^3 + 29(1 - t)^2 - 26(1 - t) + 8 \overset{?}{\geq} 0 \Leftrightarrow 7t^2 + (2t - 1)^2 \overset{?}{\geq} 0 \rightarrow \text{true}$
 $\Rightarrow$ ② is true $\therefore$ both ① and ② are true $\Rightarrow$ (**) is true whenever $\hat{A}$ is acute
 $\therefore 2(m_b^2 + m_c^2) \leq 2r_a^2 + r_b^2 + r_c^2$; with equality occurring iff $C = 1$ and $S = \frac{1}{2}$
 $\Rightarrow$ iff ΔABC is equilateral and so, $2(m_b^2 + m_c^2) \leq 2r_a^2 + r_b^2 + r_c^2 \forall \Delta ABC$,
 $" = "$ iff ΔABC is equilateral (QED)