ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta\, ABC$ with $a \ge b \ge c$ and $d \ge e \ge f \ge 0$, the following relationships hold :

$$d(3rr_a - r_a h_a) + e(3rr_b - r_b h_b) + f(3rr_c - r_c h_c) \ge 0$$
$$f(bc - 4rr_a) + e(ca - 4rr_b) + d(ab - 4rr_c) \ge 0$$

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$$3rr_a - r_a h_a \overset{?}{\ge} 3rr_b - r_b h_b$$
$$\Leftrightarrow \frac{3(s-a)(s-b)(s-c)}{s-a} - \frac{3(s-a)(s-b)(s-c)}{s-b} \overset{?}{\ge}$$
$$\frac{rs^2}{2R}\left(1+\tan^2\frac{A}{2} - 1 - \tan^2\frac{B}{2}\right) \Leftrightarrow 3(s-c)(a-b) \overset{?}{\ge} \frac{r}{2R}\cdot\frac{r^2s^2}{r^4s^2}.(s-c)^2(a-b)c$$
$$\Leftrightarrow 3 \overset{?}{\ge} \frac{c(s-c)}{2Rr} = \frac{2sc(s-c)}{abc}\ (\because a-b \ge 0) \Leftrightarrow 3(y+z)(z+x) \overset{?}{\ge} 2z(x+y+z)$$
$$\left(\begin{matrix} x = s-a, y = s-b, z = s-c \Rightarrow a = y+z, b = z+x, c = x+y \text{ and} \\ s = x+y+z \end{matrix}\right)$$
$$\Leftrightarrow 3z^2 + 3\sum_{cyc} xy \overset{?}{\ge} 2z^2 + 2\sum_{cyc} xy \Leftrightarrow z^2 + \sum_{cyc} xy \overset{?}{\ge} 0 \to \text{true (strict inequality)}$$
$$\therefore 3rr_a - r_a h_a \ge 3rr_b - r_b h_b \text{ and analogously, since : } b \ge c$$
$$\therefore 3rr_b - r_b h_b \ge 3rr_c - r_c h_c \Rightarrow 3rr_a - r_a h_a \ge 3rr_b - r_b h_b \ge 3rr_c - r_c h_c \text{ and } \because$$
$$d \ge e \ge f \ge 0 \therefore d.(3rr_a - r_a h_a) + e.(3rr_b - r_b h_b) + f.(3rr_c - r_c h_c) \overset{\text{Chebyshev}}{\ge}$$
$$\frac{1}{3}(d+e+f)\left(\sum_{cyc}(3rr_a - r_a h_a)\right) = \frac{1}{3}(d+e+f)\left(3r(4R+r) - \frac{rs^2}{2R}.\sum_{cyc}\sec^2\frac{A}{2}\right)$$
$$= \frac{1}{3}(d+e+f)\left(3r(4R+r) - \frac{rs^2}{2R}.\frac{s^2+(4R+r)^2}{s^2}\right)$$
$$= \frac{r}{6R}(d+e+f)(8R^2 - 2Rr - r^2 - s^2)$$
$$= \frac{r}{6R}(d+e+f)\left((4R^2+4Rr+3r^2-s^2) + 2(2R+r)(R-2r)\right) \ge 0$$
$$(\text{via Gerretsen and Euler and } \because d \ge e \ge f \ge 0 \Rightarrow d+e+f \ge 0)$$
$$\therefore d.(3rr_a - r_a h_a) + e.(3rr_b - r_b h_b) + f.(3rr_c - r_c h_c) \ge 0$$
$$\text{Again, Since } a \ge b \ge c, \therefore bc \le ca \le ab \text{ and } -4rr_a \le -4rr_b \le -4rr_c$$
$$\therefore bc - 4rr_a \le ca - 4rr_b \le ab - 4rr_c \text{ and } \because 0 \le f \le e \le d$$
$$\therefore f.(bc - 4rr_a) + e.(ca - 4rr_b) + d.(ab - 4rr_c) \overset{\text{Chebyshev}}{\ge}$$
$$\frac{1}{3}(d+e+f)\left(\sum_{cyc}(bc - 4rr_a)\right) = \frac{1}{3}(d+e+f)\left(s^2+4Rr+r^2-4r(4R+r)\right)$$
$$= \frac{1}{3}(d+e+f)(s^2 - 12Rr - 3r^2)$$

$$= \frac{1}{3}(d+e+f)\left((s^2-16Rr+5r^2)+4r(R-2r)\right) \geq 0$$

$(via\ \text{Gerretsen}\ and\ \text{Euler}\ and\ \because d \geq e \geq f \geq 0 \Rightarrow d+e+f \geq 0)$

$\therefore f.(bc-4rr_a)+e.(ca-4rr_b)+d.(ab-4rr_c) \geq 0$ and so,

$d.(3rr_a-r_ah_a)+e.(3rr_b-r_bh_b)+f.(3rr_c-r_ch_c) \geq 0$ and

$f.(bc-4rr_a)+e.(ca-4rr_b)+d.(ab-4rr_c) \geq 0\ \forall\ \Delta ABC$ with

$a \geq b \geq c$ and $d \geq e \geq f \geq 0$, $'' = ''$ iff ΔABC is equilateral (QED)