ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $a \ge b \ge c$ and $d \ge e \ge f \ge 0$, the following relationships hold:

$$d(h_b h_c - 3rh_a) + e(h_c h_a - 3rh_b) + f(h_a h_b - 3rh_c) \ge 0$$
$$f(3rm_a - h_b h_c) + e(3rm_b - h_c h_a) + d(3rm_c - h_a h_b) \ge 0$$

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$$\text{Since } a \ge b \ge c, \therefore h_b h_c \ge h_c h_a \ge h_a h_b \text{ and } -3rh_a \ge -3rh_b \ge -3rh_c$$
$$\therefore h_b h_c - 3rh_a \ge h_c h_a - 3rh_b \ge h_a h_b - 3rh_c \text{ and } \because d \ge e \ge f \ge 0$$
$$\therefore d.(h_b h_c - 3rh_a) + e.(h_c h_a - 3rh_b) + f.(h_a h_b - 3rh_c) \overset{\text{Chebyshev}}{\ge}$$
$$\frac{1}{3}(d+e+f)\left(\sum_{cyc}(h_b h_c - 3rh_a)\right) = \frac{1}{3}(d+e+f)\left(\frac{2rs^2}{R} - \frac{3r}{2R}.\sum_{cyc} ab\right)$$
$$= \frac{1}{3}(d+e+f)\left(\frac{r}{2R}\right)\left(\left(\sum_{cyc} a\right)^2 - 3\sum_{cyc} ab\right) \ge 0$$
$$(\because d \ge e \ge f \ge 0 \Rightarrow d+e+f \ge 0)$$
$$\therefore d.(h_b h_c - 3rh_a) + e.(h_c h_a - 3rh_b) + f.(h_a h_b - 3rh_c) \ge 0$$
$$\text{Again, Since } a \ge b \ge c, \therefore -h_b h_c \le -h_c h_a \le -h_a h_b \text{ and } 3rm_a \le 3rm_b \le 3rm_c$$
$$\therefore 3rm_a - h_b h_c \le 3rm_b - h_c h_a \le 3rm_c - h_a h_b \text{ and } \because 0 \le f \le e \le d$$
$$\therefore f.(3rm_a - h_b h_c) + e.(3rm_b - h_c h_a) + d.(3rm_c - h_a h_b) \overset{\text{Chebyshev}}{\ge}$$
$$\frac{1}{3}(d+e+f)\left(\sum_{cyc}(3rm_a - h_b h_c)\right) \overset{\text{Tereshin}}{\ge} \frac{1}{3}(d+e+f)\left(3r\sum_{cyc}\frac{b^2+c^2}{4R} - \frac{2rs^2}{R}\right)$$
$$= \frac{1}{3}(d+e+f)\left(\frac{r}{2R}\right)\left(3\sum_{cyc} a^2 - \left(\sum_{cyc} a\right)^2\right) \ge 0$$
$$(\because d \ge e \ge f \ge 0 \Rightarrow d+e+f \ge 0)$$
$$\therefore f.(3rm_a - h_b h_c) + e.(3rm_b - h_c h_a) + d.(3rm_c - h_a h_b) \ge 0 \text{ and so,}$$
$$d.(h_b h_c - 3rh_a) + e.(h_c h_a - 3rh_b) + f.(h_a h_b - 3rh_c) \ge 0 \text{ and}$$
$$f.(3rm_a - h_b h_c) + e.(3rm_b - h_c h_a) + d.(3rm_c - h_a h_b) \ge 0 \ \forall \ \Delta \text{ ABC with}$$
$$a \ge b \ge c \text{ and } d \ge e \ge f \ge 0, '' = '' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$