ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $a \geq b \geq c$ and $d \geq e \geq f \geq 0$, the following relationship holds :

$$d(3Rm_a - 2r_br_c) + e(3Rm_b - 2r_cr_a) + f(3Rm_c - 2r_ar_b) \geq 0$$

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Since $d \geq e \geq f \geq 0 \therefore$ we can assign :

$\boxed{e = f + g}$ and $\boxed{d = f + g + h}$ $(g, h \geq 0)$ & via such substitutions,

$$d.(3Rm_a - 2r_br_c) + e.(3Rm_b - 2r_cr_a) + f.(3Rm_c - 2r_ar_b) \overset{?}{\geq} 0$$

$$\Leftrightarrow f.\left(3R\sum_{cyc} m_a - 2s\sum_{cyc}(s-a)\right) + g.\big(3R(m_a + m_b) - 2s\big((s-a)+(s-b)\big)\big) +$$

$$h.\big(3Rm_a - 2s(s-a)\big) \boxed{\overset{?}{\underset{(*)}{\geq}}} 0$$

$$\text{Firstly}, 3R(m_a + m_b) - 2s\big((s-a)+(s-b)\big) \overset{\text{Tereshin}}{\geq} \frac{3(b^2 + 2c^2 + a^2)}{4} - 2sc \overset{?}{\geq} 0$$

$$\Leftrightarrow 3((z+x)^2 + 2(x+y)^2 + (y+z)^2) \overset{?}{\geq} 8(x+y+z)(x+y)$$

$$\left(\begin{matrix} x = s-a, y = s-b, z = s-c \Rightarrow a = y+z, b = z+x, c = x+y \text{ and} \\ s = x+y+z \end{matrix}\right)$$

$$\Leftrightarrow x^2 - 4xy - 2zx + y^2 - 2yz + 6z^2 \overset{?}{\geq} 0 \Leftrightarrow (y-z)^2 + (z-x)^2 + 4(z^2 - xy) \overset{?}{\geq} 0$$

$$\to \text{true} \because z^2 \overset{?}{\geq} xy (\because a \geq b \geq c \Rightarrow z \geq y \geq x)$$

$$\therefore 3R(m_a + m_b) - 2s\big((s-a)+(s-b)\big) \boxed{\overset{①}{\geq}} 0$$

$$\text{Now}, (3Rm_a)^2 \overset{\substack{\text{Leibnitz} \\ \text{and} \\ \text{Lascu + AM-GM}}}{\geq} \left(\sum_{cyc} a^2\right) s(s-a) \overset{?}{\geq} 4s^2(s-a)^2$$

$$\Leftrightarrow (y+z)^2 + (z+x)^2 + (x+y)^2 \overset{?}{\geq} 4x(x+y+z)$$

$$\Leftrightarrow (y^2 - x^2) + y(z-x) + z(z-x) \overset{?}{\geq} 0 \to \text{true} \because z \geq y \geq x$$

$$\therefore 3Rm_a - 2s(s-a) \boxed{\overset{②}{\geq}} 0 \text{ and finally}, 3R\sum_{cyc} m_a - 2s\sum_{cyc}(s-a) \overset{\text{Tereshin}}{\geq}$$

$$\frac{3}{2}.\sum_{cyc} a^2 - 2s^2 = 3\sum_{cyc} a^2 - \left(\sum_{cyc} a\right)^2 \geq 0 \therefore 3R\sum_{cyc} m_a - 2s\sum_{cyc}(s-a) \boxed{\overset{③}{\geq}} 0$$

$\therefore$ ①, ② and ③ combined with $f, g, h \geq 0 \Rightarrow (*)$ is true

$\therefore d.(3Rm_a - 2r_br_c) + e.(3Rm_b - 2r_cr_a) + f.(3Rm_c - 2r_ar_b) \geq 0 \ \forall \Delta$ ABC with $a \geq b \geq c$ and $d \geq e \geq f \geq 0$, " = " iff Δ ABC is equilateral (QED)