ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{s}{w_a} \le \frac{w_a}{s} + \frac{a}{h_a}$$

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Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$

$$c = x + y \text{ and } s = x + y + z \therefore \frac{s}{w_a} \overset{?}{\le} \frac{w_a}{s} + \frac{a}{h_a} \Leftrightarrow \frac{\left(s^2 - bc + \frac{a^2bc}{(b+c)^2}\right)^2}{s^2\left(bc - \frac{a^2bc}{(b+c)^2}\right)}$$

$$\overset{?}{\le} \frac{a^4}{4s(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow (y+z)^4(2x+y+z)^2(x+y+z)^2\left(\begin{matrix}(z+x)(x+y)(2x+y+z)^2 - \\ (z+x)(x+y)(y+z)^2\end{matrix}\right) \overset{?}{\ge}$$

$$4xyz(x+y+z)\left(\begin{matrix}(x+y+z)^2(2x+y+z)^2 - (z+x)(x+y)(2x+y+z)^2 + \\ (z+x)(x+y)(y+z)^2\end{matrix}\right)^2$$

$$\Leftrightarrow 4x^6(y^2-z^2)^2 + x^5.\overbrace{\left(20(y^5+z^5) + 12(y^4z+yz^4) - 32(y^3z^2+y^2z^3)\right)}^{\sigma_1} +$$

$$x^4.\overbrace{\left(41(y^6+z^6) + 49(y^5z+yz^5) - 37(y^4z^2+y^2z^4) - 106y^3z^3\right)}^{\sigma_2} +$$

$$x^3.\overbrace{\left(44(y^7+z^7) + 79(y^6z+yz^6) + 3(y^5z^2+y^2z^5) - 126(y^4z^3+y^3z^4)\right)}^{\sigma_3} +$$

$$x^2.\overbrace{\left(26(y^8+z^8) + 63(y^7z+yz^7) + 34(y^6z^2+y^2z^6) - 63(y^5z^3+y^3z^5) - 120y^4z^4\right)}^{\sigma_4}$$

$$+x.\overbrace{\left(8(y^9+z^9) + 25(y^8z+yz^8) + 23(y^7z^2+y^2z^7) - 11(y^6z^3+y^3z^6) - 45(y^5z^4+y^4z^5)\right)}^{\sigma_5}$$

$$+\overbrace{y^{10} + z^{10} + 4(y^9z+yz^9) + 5(y^8z^2+y^2z^8) - 6(y^6z^4+y^4z^6) - 8y^5z^5}^{\sigma_6} \boxed{\underset{(*)}{\overset{?}{\ge}}} 0$$

- $$\left\{\begin{matrix} 20\left(y^5+z^5-(y^3z^2+y^2z^3)\right) = 20(y-z)^2(y+z)(y^2+yz+z^2) \ge 0 \text{ and} \\ 12\left(y^4z+yz^4-(y^3z^2+y^2z^3)\right) = 12yz(y-z)^2(y+z) \ge 0 \Rightarrow \sigma_1 \ge 0 \end{matrix}\right\}$$

- $$\left\{\begin{matrix} 37\left(y^5z+yz^5-(y^4z^2+y^2z^4)\right) = 37yz(y-z)^2(y^2+yz+z^2) \ge 0 \text{ and} \\ 41(y^6+z^6) + 12(y^5z+yz^5) \overset{\text{AM-GM}}{\ge} 106y^3z^3 \Rightarrow \sigma_2 \ge 0 \end{matrix}\right\}$$

- $$\left\{\begin{matrix} 44\left(y^7+z^7-(y^4z^3+y^3z^4)\right) = 44(y-z)^2(y+z)(y^2+yz+z^2)(y^2+z^2) \ge 0 \text{ and} \\ 79\left(y^6z+yz^6-(y^4z^3+y^3z^4)\right) = 79yz(y-z)^2(y+z)(y^2+yz+z^2) \ge 0 \text{ and} \\ 3\left(y^5z^2+y^2z^5-(y^4z^3+y^3z^4)\right) = 3y^2z^2(y-z)^2(y+z) \ge 0 \Rightarrow \sigma_3 \ge 0 \end{matrix}\right\}$$

- $$\begin{Bmatrix} 63\left(y^7z + yz^7 - \left(y^5z^3 + y^3z^5\right)\right) = 63yz(y^2 - z^2)^2(y^2 + z^2) \geq 0 \; and \\ 26(y^8 + z^8) + 34(y^6z^2 + y^2z^6) \overset{AM-GM}{\geq} 120y^4z^4 \Rightarrow \sigma_4 \geq 0 \end{Bmatrix}$$

- $$\begin{Bmatrix} 8\left(y^9 + z^9 - (y^6z^3 + y^3z^6)\right) = 8(y^3 - z^3)^2(y^3 + z^3) \geq 0 \; and \\ 3\left(y^8z + yz^8 - (y^6z^3 + y^3z^6)\right) = 3yz(y - z)^2(y + z)(y^4 + y^3z + y^2z^2 + yz^3 + z^4) \geq 0 \; and \\ 22\left(y^8z + yz^8 - \left(y^5z^4 + y^4z^5\right)\right) = 22yz(y - z)^2(y^2 + yz + z^2)(y + z)(y^2 + z^2) \geq 0 \; and \\ 23\left(y^7z^2 + y^2z^7 - \left(y^5z^4 + y^4z^5\right)\right) = 23y^2z^2(y - z)^2(y + z)(y^2 + yz + z^2) \geq 0 \Rightarrow \sigma_5 \geq 0 \end{Bmatrix}$$

- $$\begin{Bmatrix} y^{10} + z^{10} - (y^6z^4 + y^4z^6) = (y^2 - z^2)^2(y^2 + z^2)(y^4 + y^2z^2 + z^4) \geq 0 \; and \\ 4\left(y^9z + yz^9 - (y^6z^4 + y^4z^6)\right) = 4yz(y - z)^2(y^2 + yz + z^2)(y^4 + y^3z + y^2z^2 + yz^3 + z^4) \\ \geq 0 \text{ and } y^8z^2 + y^2z^8 - (y^6z^4 + y^4z^6) = y^2z^2(y^2 - z^2)^2(y^2 + z^2) \geq 0 \\ and \; 4(y^8z^2 + y^2z^8) \overset{AM-GM}{\geq} 8y^5z^5 \Rightarrow \sigma_6 \geq 0 \end{Bmatrix}$$

$$\therefore \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6 \geq 0 \Rightarrow (*) \text{ is true } \therefore \frac{s}{w_a} \leq \frac{w_a}{s} + \frac{a}{h_a} \; \forall \, \Delta \, ABC,$$
$$'' = '' \text{ iff } b = c \text{ (QED)}$$