ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $a \ge b \ge c$ and $d \ge e \ge f \ge 0$, the following relationships hold :

$$d\left(r_a - \frac{\sqrt{3}}{2}a\right) + e\left(r_b - \frac{\sqrt{3}}{2}b\right) + f\left(r_c - \frac{\sqrt{3}}{2}c\right) \ge 0$$

$$f\left(r_b + r_c - \frac{\sqrt{3}(b+c)}{2}\right) + e\left(r_c + r_a - \frac{\sqrt{3}(c+a)}{2}\right) + d\left(r_a + r_b - \frac{\sqrt{3}(a+b)}{2}\right) \ge 0$$

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$$r_a - \frac{\sqrt{3}}{2}a \overset{?}{\ge} r_b - \frac{\sqrt{3}}{2}b \Leftrightarrow \frac{rs}{s-a} - \frac{rs}{s-b} \underbrace{\overset{?}{\ge}}_{(*)} \frac{\sqrt{3}}{2}(a-b)$$

$$\Leftrightarrow \frac{rs(a-b)(s-c)}{r^2 s} \overset{?}{\ge} \frac{\sqrt{3}}{2}(a-b) \Leftrightarrow \frac{(a-b)^2(s-c)^2 s}{(s-a)(s-b)(s-c)} \overset{?}{\ge} \frac{3}{4}(a-b)^2 \ (\because a \ge b)$$

$$\Leftrightarrow 4(x+y+z)z \overset{?}{\ge} 3xy \ \left(\begin{matrix} x = s-a, y = s-b, z = s-c \Rightarrow a = y+z, b = z+x, \\ c = x+y \text{ and } s = x+y+z \end{matrix}\right)$$

$$\Leftrightarrow 4z^2 + 4zx + 4yz \overset{?}{\ge} 3xy \to \text{true (strict inequality)} \because 3z^2 \overset{?}{\ge} 3xy$$

$$(\because a \ge b \ge c \Rightarrow z \ge y \ge x) \therefore r_a - \frac{\sqrt{3}}{2}a \ge r_b - \frac{\sqrt{3}}{2}b \text{ and analogously,}$$

$$r_b - \frac{\sqrt{3}}{2}b \ge r_c - \frac{\sqrt{3}}{2}c \therefore r_a - \frac{\sqrt{3}}{2}a \ge r_b - \frac{\sqrt{3}}{2}b \ge r_c - \frac{\sqrt{3}}{2}c \text{ and } \because d \ge e \ge f \ge 0$$

$$\therefore d.\left(r_a - \frac{\sqrt{3}}{2}a\right) + e.\left(r_b - \frac{\sqrt{3}}{2}b\right) + f.\left(r_c - \frac{\sqrt{3}}{2}c\right) \overset{\text{Chebyshev}}{\ge}$$

$$\frac{1}{3}(d+e+f)\left(\sum_{cyc}\left(r_a - \frac{\sqrt{3}}{2}a\right)\right) = \frac{1}{3}(d+e+f)\left(4R + r - \frac{\sqrt{3}}{2}.2s\right) \overset{\text{Doucet or Trucht}}{\ge} 0$$

$$(\because d \ge e \ge f \ge 0 \Rightarrow d+e+f \ge 0)$$

$$\therefore \boxed{d.\left(r_a - \frac{\sqrt{3}}{2}a\right) + e.\left(r_b - \frac{\sqrt{3}}{2}b\right) + f.\left(r_c - \frac{\sqrt{3}}{2}c\right) \ge 0} \text{ and again,}$$

$$r_b + r_c - \frac{\sqrt{3}(b+c)}{2} \overset{?}{\le} r_c + r_a - \frac{\sqrt{3}(c+a)}{2} \Leftrightarrow \frac{rs}{s-a} - \frac{rs}{s-b} \overset{?}{\ge} \frac{\sqrt{3}}{2}(a-b) \to \text{true}$$

$$\because \text{it's inequality } (*) \therefore r_b + r_c - \frac{\sqrt{3}(b+c)}{2} \le r_c + r_a \text{ and analogously,}$$

$$r_c + r_a - \frac{\sqrt{3}(c+a)}{2} \le r_a + r_b - \frac{\sqrt{3}(a+b)}{2}$$

$$\therefore r_b + r_c - \frac{\sqrt{3}(b+c)}{2} \le r_c + r_a - \frac{\sqrt{3}(c+a)}{2} \le r_a + r_b - \frac{\sqrt{3}(a+b)}{2} \text{ and } \because$$

$$0 \le f \le e \le d \therefore f\left(r_b + r_c - \frac{\sqrt{3}(b+c)}{2}\right) + e.\left(r_c + r_a - \frac{\sqrt{3}(c+a)}{2}\right) +$$

$$d.\left(r_a+r_b-\frac{\sqrt{3}(a+b)}{2}\right)\overset{\text{Chebyshev}}{\geq}\frac{1}{3}(d+e+f)\left(\sum_{\text{cyc}}\left(r_b+r_c-\frac{\sqrt{3}(b+c)}{2}\right)\right)$$

$$=\frac{1}{3}(d+e+f)\left(2(4R+r)-\sum_{\text{cyc}}\frac{\sqrt{3}(b+c)}{2}\right)=\frac{1}{3}(d+e+f)\big(2(4R+r)-\sqrt{3}.2s\big)$$

$$\overset{\text{Doucet or Trucht}}{\geq}0\ (\because d\geq e\geq f\geq 0\Rightarrow d+e+f\geq 0)$$

$$\therefore \boxed{f.\left(r_b+r_c-\frac{\sqrt{3}(b+c)}{2}\right)+e.\left(r_c+r_a-\frac{\sqrt{3}(c+a)}{2}\right)+d.\left(r_a+r_b-\frac{\sqrt{3}(a+b)}{2}\right)\geq 0}$$

$$and\ \text{so}, d.\left(r_a-\frac{\sqrt{3}}{2}a\right)+e.\left(r_b-\frac{\sqrt{3}}{2}b\right)+f.\left(r_c-\frac{\sqrt{3}}{2}c\right)\geq 0\ and$$

$$f\left(r_b+r_c-\frac{\sqrt{3}(b+c)}{2}\right)+e\left(r_c+r_a-\frac{\sqrt{3}(c+a)}{2}\right)+d\left(r_a+r_b-\frac{\sqrt{3}(a+b)}{2}\right)\geq 0$$

$\forall\ \Delta\ ABC$ with $a\geq b\geq c$ and $d\geq e\geq f\geq 0$, $''=''$ iff $\Delta\ ABC$ is equi$lateral$ (QED)