ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$m_a^2+\frac{1}{2}r_a(r_b+r_c+2h_a)\le s^2\le\frac{1}{2}(m_b+m_c)^2+\frac{1}{4}(r_b+r_c)^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava chakraborty-Kolkata-India

$$2m_bm_c\overset{?}{\ge}\frac{9}{4}a^2+h_a^2-m_b^2-m_c^2=\frac{9a^2-4a^2-b^2-c^2}{4}+\frac{4F^2}{a^2}$$

$$=\frac{a^2(5a^2-b^2-c^2)+2\sum_{cyc}a^2b^2-\sum_{cyc}a^4}{4a^2}=\frac{a^2(4a^2+b^2+c^2)-(b^2-c^2)^2}{4a^2}$$

and it's trivially true if : $a^2(4a^2+b^2+c^2)-(b^2-c^2)^2\le 0$ and when : $a^2(4a^2+b^2+c^2)-(b^2-c^2)^2>0$, then we are to prove :

$$4a^4(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)\overset{?}{\ge}\left(a^2(4a^2+b^2+c^2)-(b^2-c^2)^2\right)^2$$

$$\Leftrightarrow -a^4(b^2-c^2)^2+2a^2(b^2+c^2)(b^2-c^2)^2-(b^2-c^2)^4\overset{?}{\ge}0\Leftrightarrow$$

$$(b^2-c^2)^2(-a^4+2a^2(b^2+c^2)-b^4-c^4+2b^2c^2)\overset{?}{\ge}0\Leftrightarrow(b^2-c^2)^2.16F^2\overset{?}{\ge}0$$

$$\to \text{true}\ \therefore\ 2m_bm_c\ge\frac{9}{4}a^2+h_a^2-m_b^2-m_c^2\Rightarrow\frac{1}{2}(m_b+m_c)^2+\frac{1}{4}(r_b+r_c)^2\ge$$

$$\frac{9}{8}a^2+\frac{h_a^2}{2}+\frac{1}{4}(r_b+r_c)^2\overset{?}{\ge}s^2$$

$$\Leftrightarrow\frac{9}{8}.16R^2S^2(1-S^2)+\frac{16R^4(C^2-S^2)^2}{8R^2}+4R^2(1-S^2)^2\overset{?}{\ge}4R^2(1-S^2)(C+S)^2$$

$$\left(S=\sin\frac{A}{2},C=\cos\frac{B-C}{2}\right)\Leftrightarrow(C^2-1)^2+1-4S^4+3S^2-4S(1-S^2)C\overset{?}{\ge}0\ \&\ \because$$

$$(C^2-1)^2\ge 0\ \text{and}\ -4S(1-S^2)C\ge-4S(1-S^2)\ (\because C\le 1\ \text{and}\ S^2<1)$$

$$\therefore\ \text{it suffices to prove : }1-4S^4+3S^2-4S(1-S^2)\overset{?}{\ge}0\Leftrightarrow\boxed{(1-S^2)(2S-1)^2\overset{?}{\ge}0}$$

$$\to\text{true}\ \therefore\ \boxed{s^2\le\frac{1}{2}(m_b+m_c)^2+\frac{1}{4}(r_b+r_c)^2}$$

$$\text{Again, }m_a^2+\frac{1}{2}r_a(r_b+r_c+2h_a)\overset{?}{\le}s^2$$

$$\Leftrightarrow 4s^2-(4s(s-a)+(b-c)^2)\overset{?}{\ge}2s^2-2s(s-a)+4r_ah_a$$

$$\Leftrightarrow 2sa\overset{?}{\ge}(b-c)^2+4r_ah_a$$

$$\Leftrightarrow 16R^2(1-S^2)(C+S)S\overset{?}{\ge}16R^2S^2(1-C^2)+\frac{8RS(C+S)}{2R}.2R^2(2C^2-2S^2)$$

$$\Leftrightarrow\boxed{C(1-C^2)\overset{?}{\ge}0}\to\text{true}\because 0<C\le 1\ \therefore\ \boxed{m_a^2+\frac{1}{2}r_a(r_b+r_c+2h_a)\le s^2}$$

and so, $m_a^2+\frac{1}{2}r_a(r_b+r_c+2h_a)\le s^2\le\frac{1}{2}(m_b+m_c)^2+\frac{1}{4}(r_b+r_c)^2\ \forall\ \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)