ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$2m_a + 2r_a + r_b + r_c \ge 2\sqrt{3}s$$

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$$\text{For own convenience, semi} - \text{perimeter} = s_0 \text{ and now,}$$

$$2m_a + 2r_a + r_b + r_c \overset{\text{Lascu}}{\ge} 4R(1-s^2).c + 4R.s(c+s) + 4R(1-s^2)$$

$$\left(s = \sin\frac{A}{2}, c = \cos\frac{B-C}{2}\right) \overset{?}{\ge} 2\sqrt{3}.s = \sqrt{3}.4R.\sqrt{1-s^2}.(c+s)$$

$$\Leftrightarrow \left(1+s-s^2-\sqrt{3-3s^2}\right)c + 1 - s.\sqrt{3-3s^2} \boxed{\overset{?}{\underset{(*)}{\ge}}} 0$$

$$\text{Now, discriminant of : } 3s^4 - 3s^2 + 1 = 9 - 12 < 0 \Rightarrow 3s^4 - 3s^2 + 1 > 0$$

$$\Rightarrow 1 > 3s^2(1-s^2) \Rightarrow 1 > s.\sqrt{3-3s^2}\ (\because 0 < s < 1) \text{ and so,}$$

$$\boxed{\text{if : } 1+s-s^2-\sqrt{3-3s^2} \ge 0}, \text{then : } \boxed{\boxed{\text{LHS of } (*) > 0}} \text{ and}$$

$$\boxed{\text{when : } 1+s-s^2-\sqrt{3-3s^2} < 0}, \text{then : } \left(1+s-s^2-\sqrt{3-3s^2}\right)c \ge$$

$$1+s-s^2-\sqrt{3-3s^2}\ (\because c \le 1) \text{ and so, in order to prove } (*), \text{it suffices to prove :}$$

$$1+s-s^2-\sqrt{3-3s^2}+1-s.\sqrt{3-3s^2} \overset{?}{\ge} 0 \Leftrightarrow 2+s-s^2 \overset{?}{\ge} \sqrt{3-3s^2}.(1+s)$$

$$\Leftrightarrow (1+s)(2-s) \overset{?}{\ge} \sqrt{3-3s^2}.(1+s) \Leftrightarrow (2-s)^2 \overset{?}{\ge} 3-3s^2 \Leftrightarrow 4s^2-4s+1 \overset{?}{\ge} 0$$

$$\Leftrightarrow (2s-1)^2 \overset{?}{\ge} 0 \to \text{true} \Rightarrow \boxed{\boxed{(*) \text{ is true}}} \text{ and so, } 2m_a + 2r_a + r_b + r_c \ge 2\sqrt{3}.s$$

$$\forall\, \Delta\, \text{ABC}, '' = '' \text{ iff } \Delta\, \text{ABC is equilateral (QED)}$$