ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC the following relationship holds :

$$\frac{1}{m_a}+\frac{4}{r_a+m_a}\ge\frac{2}{R}$$

*Proposed by Dang Ngoc Minh-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\boxed{\text{Case 1}}\ m(\angle A)<\frac{\pi}{2}\text{ and then : }\frac{1}{m_a}+\frac{4}{r_a+m_a}\ge$$

$$\frac{1}{2R\cos^2\frac{A}{2}}+\frac{4}{4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+2R\cos^2\frac{A}{2}}$$

$$=\frac{1}{2R\cos^2\frac{A}{2}}+\frac{4}{\left(2R\sin\frac{A}{2}\right)\left(\sin\frac{A}{2}+\cos\frac{B-C}{2}\right)+2R\cos^2\frac{A}{2}}=\frac{1}{2R(1-s^2)}+\frac{1}{2R(1+cs)}$$

$$\begin{pmatrix}c=\cos\frac{B-C}{2}\\ s=\sin\frac{A}{2}\end{pmatrix}\overset{?}{\ge}\frac{2}{R}\Leftrightarrow 1\overset{?}{\ge}c(3s-4s^3)\ \&\ \because m(\angle A)<\frac{\pi}{2}\Rightarrow s<\frac{1}{\sqrt{2}}\therefore s^2<\frac{1}{2}<\frac{3}{4}$$

$$\Rightarrow 3s-4s^3>0\text{ and so, }c(3s-4s^3)\le 3s-4s^3;\text{ hence it suffices to prove :}$$

$$1\overset{?}{\ge}3s-4s^3\Leftrightarrow(s+1)(2s-1)^2\overset{?}{\ge}0\to\text{true}\therefore\frac{1}{m_a}+\frac{4}{r_a+m_a}\ge\frac{2}{R}$$

$$\boxed{\text{Case 2}}\ m(\angle A)\ge\frac{\pi}{2}\text{ and then : }\frac{1}{m_a}+\frac{4}{r_a+m_a}\ge\frac{2}{a}+\frac{8}{2r_a+a}=$$

$$\frac{2}{4R.s.\sqrt{1-s^2}}+\frac{8}{4R.s(s+c)+4R.s.\sqrt{1-s^2}}\overset{?}{\ge}\frac{2}{R}\Leftrightarrow\frac{1}{4su}+\frac{1}{s^2+sc+su}\overset{?}{\ge}1$$

$$\left(u=\sqrt{1-s^2}\right)\text{ and }\because c\le 1;\text{ hence it suffices to prove : }\frac{1}{4su}+\frac{1}{s^2+s+su}\overset{?}{\ge}1$$

$$\Leftrightarrow -4s^2u-4su^2-4su+s+5u+1\overset{?}{\ge}0$$

$$\Leftrightarrow -4s^2u-4s(1-s^2)-4su+s+5u+1\overset{?}{\ge}0$$

$$\Leftrightarrow 4s^3-3s+1\underbrace{\overset{?}{\ge}}_{(*)}(4s^2+4s-5).\sqrt{1-s^2}$$

$$\boxed{\text{Case 2}(a)}\ 4s^2+4s-5\le 0\text{ and then : LHS of }(*)=(s+1)(2s-1)^2\overset{?}{\ge}0$$

$$\ge\text{RHS of }(*)\text{ (strict inequality)}$$

$$\boxed{\text{Case 2(b)}}\ 4s^2+4s-5>0\text{ and then : }(*)\Leftrightarrow$$

$$(4s^3-3s+1)^2\overset{?}{\ge}(1-s^2)(4s^2+4s-5)^2$$

$$\Leftrightarrow\frac{2(s+1)\left(\left((500s+300)(4s^2+4s-5)+1140s-804\right)(5s-4)^2+1025s-636\right)}{3125}\overset{?}{\ge}0$$

$$\rightarrow \text{true (strict inequality)} \because 4s^2 + 4s - 5 > 0 \text{ and } \because s \geq \frac{1}{\sqrt{2}} \left(\because m(\angle A) \geq \frac{\pi}{2}\right)$$

$$\Rightarrow 1140s > 806 > 804 \text{ and } 1025s > 724 > 636 \therefore \text{combining cases } 2(a) \,\&\, 2(b),$$

$$\frac{1}{m_a} + \frac{4}{r_a + m_a} > \frac{2}{R} \text{ whenever } m(\angle A) \geq \frac{\pi}{2} \text{ and combining cases 1 and 2,}$$

$$\frac{1}{m_a} + \frac{4}{r_a + m_a} \geq \frac{2}{R} \; \forall \, \Delta \, ABC, '' = '' \text{ iff } \Delta \, ABC \text{ is equilateral (QED)}$$