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In any Δ ABC the following relationship holds :

$$\frac{1}{2r_a+R}+\frac{3\sqrt{3}}{2(b+c)}\ge\frac{1}{R}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{2r_a+R}+\frac{3\sqrt{3}}{2(b+c)}\overset{?}{\ge}\frac{1}{R}\Leftrightarrow\frac{3\sqrt{3}}{2(b+c)}\underset{(*)}{\overset{?}{\ge}}\frac{2r_a}{R(2r_a+R)}$$

$$\text{Now, }\frac{2r_a}{R(2r_a+R)}\overset{\text{Euler}}{\le}\frac{2r_a}{R(2r_a+2r)}=\frac{\frac{rs}{s-a}}{R\left(\frac{rs}{s-a}+\frac{rs}{s}\right)}=\frac{\frac{rs}{s-a}.s(s-a)}{R.rs.(b+c)}=\frac{s}{R(b+c)}$$

$$\overset{\text{Mitrinovic}}{\le}\frac{3\sqrt{3}}{2(b+c)}\Rightarrow(*)\text{ is true}\therefore\frac{1}{2r_a+R}+\frac{3\sqrt{3}}{2(b+c)}\ge\frac{1}{R}\ \forall\,\Delta\text{ ABC},$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$