ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta\, ABC$ with $a \geq b \geq c$ and $0 \leq d \leq e \leq f$ the following relationship holds:

$$d(r_b + r_c - 2m_a) + e(r_c + r_a - 2m_b) + f(r_a + r_b - 2m_c) \geq 0$$

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$$\text{We shall first prove}: b(s-b) \boxed{\underset{①}{\overset{?}{\geq}}} 2r.m_b \text{ and } c(s-c) \boxed{\underset{②}{\overset{?}{\geq}}} 2r.m_c;$$

$$\text{whenever}: a \geq b \geq c; \text{Let } x = s-a, y = s-b, z = s-c \text{ and then}: a = y+z,$$
$$b = z+x, c = x+y \text{ and } s = x+y+z \text{ and so}, a \geq b \geq c \Rightarrow$$
$$y+z \geq z+x \geq x+y \Rightarrow z \geq y \geq x \text{ and so, we can assign}:$$
$$\boxed{y = x+g} \text{ and } z = \boxed{x+g+h}\ (g,h \geq 0)$$
$$\text{Also}, d \leq e \leq f \Rightarrow \text{we can assign}: \boxed{e = d+m} \text{ and } \boxed{f = d+m+n}\ (m,n \geq 0)$$
$$\text{Now}, ① \Leftrightarrow (z+x)^2y^2 \overset{?}{\geq} \frac{xyz}{x+y+z}.(4y(x+y+z)+(z-x)^2)$$
$$\Leftrightarrow (x+g+h+x)^2(x+g)(x+x+g+x+g+h) \overset{?}{\geq}$$
$$x(x+g+h)(4(x+g)(x+x+g+x+g+h)+(g+h)^2)$$
$$\Leftrightarrow 2g^4 + 5g^3h + 4g^3x + 4g^2h^2 + 8g^2hx + 2g^2x^2 + gh^3 + 4gh^2x + 4ghx^2 + 2h^2x^2$$
$$\overset{?}{\geq} 0 \to \text{true} \because g,h \geq 0 \Rightarrow \boxed{① \text{ is true}} \text{ and again}, ② \Leftrightarrow$$
$$(x+y)^2z^2 \overset{?}{\geq} \frac{xyz}{x+y+z}.(4z(x+y+z)+(x-y)^2)$$
$$\Leftrightarrow (x+x+g)^2(x+g+h)(x+x+g+x+g+h) \overset{?}{\geq}$$
$$x(x+g)(4(x+g+h)(x+x+g+x+g+h)+g^2)$$
$$\Leftrightarrow 2g^4 + 3g^3h + 4g^3xg^2h^2 + 4g^2hx + 2g^2x^2 \overset{?}{\geq} 0 \to \text{true} \because g,h \geq 0 \Rightarrow \boxed{② \text{ is true}}$$
$$\text{We have}: r_b + r_c - 2m_a = \frac{rs.a(s-a)}{r^2s} - 2m_a = \frac{a(s-a) - 2r.m_a}{r} \text{ and analogs}$$
$$\therefore \text{proposed inequality becomes}: d(a(s-a) - 2r.m_a) +$$
$$(d+m)(b(s-b) - 2r.m_b) + (d+m+n)(c(s-c) - 2r.m_c) \overset{?}{\geq} 0$$
$$\Leftrightarrow d(x(y+z) - 2r.m_a) + (d+m)(y(z+x) - 2r.m_b) +$$
$$(d+m+n)(z(x+y) - 2r.m_c) \overset{?}{\geq} 0$$
$$\Leftrightarrow 2d\left(\sum_{cyc} xy - r\left(\sum_{cyc} m_a\right)\right) + m\Big(\big((xy+yz) - 2r.m_b\big) + \big((xz+yz) - 2r.m_c\big)\Big)$$
$$+n\big((xz+yz) - 2r.m_c\big) \overset{?}{\geq} 0$$

$$\Leftrightarrow 2d\left(r(4R+r)-r\left(\sum_{cyc} m_a\right)\right)+m\big((b(s-b)-2r.m_b)+(c(s-c)-2r.m_c)\big)$$

$$+n(c(s-c)-2r.m_c)\overset{?}{\geq} 0 \rightarrow \text{true via ① } and \text{ ② combined with :}$$

$$\sum_{cyc} m_a \leq 4R+r \; and \because d,m,n \geq 0 \; and \text{ so,}$$

$$d.(r_b+r_c-2m_a)+e.(r_c+r_a-2m_b)+f.(r_a+r_b-2m_c) \geq 0$$

$$\forall \Delta\, ABC \text{ with } a \geq b \geq c \; and \; 0 \leq d \leq e \leq f, '' = '' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$