ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $a \geq b \geq c$ and

$0 \leq d \leq e \leq f$ the following relationship holds :

$$d(4r_b r_c - 3bc) + e(4r_c r_a - 3ca) + f(4r_a r_b - 3ab) \geq 0$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Charaborty-Kolkata-India

$$4r_b r_c - 3bc \overset{?}{\leq} 4r_c r_a - 3ca \Leftrightarrow 3c(a-b) \overset{?}{\leq} 4s\big((s-b)-(s-a)\big)$$

$$\Leftrightarrow (4(s-c)+c)(a-b) \overset{?}{\geq} 0 \rightarrow \text{true} \because a \geq b \therefore 4r_b r_c - 3bc \boxed{\overset{1}{\leq}} 4r_c r_a - 3ca$$

$$\text{Also}, 4r_c r_a - 3ca \overset{?}{\leq} 4r_a r_b - 3ab \Leftrightarrow 3a(b-c) \overset{?}{\leq} 4s\big((s-c)-(s-b)\big)$$

$$\Leftrightarrow (4(s-a)+a)(b-c) \overset{?}{\geq} 0 \rightarrow \text{true} \because b \geq c \therefore 4r_c r_a - 3ca \boxed{\overset{1}{\leq}} 4r_a r_b - 3ab$$

$$\therefore (1) \text{ and } (2) \Rightarrow d \leq e \leq f \text{ and } 4r_b r_c - 3bc \leq 4r_c r_a - 3ca \leq 4r_a r_b - 3ab$$

$$\therefore d.(4r_b r_c - 3bc) + e.(4r_c r_a - 3ca) + f.(4r_a r_b - 3ab) \overset{\text{Chebyshev}}{\geq}$$

$$\frac{1}{3}(d+e+f).\sum_{\text{cyc}}(4s(s-a) - 3bc) = \frac{1}{3}(d+e+f).\big(4s^2 - 3(s^2 + 4Rr + r^2)\big)$$

$$= \frac{1}{3}(d+e+f).\big(s^2 - 16Rr + 5r^2 + 4r(R-2r)\big) \overset{\substack{\text{Gerretsen} \\ \text{and} \\ \text{Euler}}}{\geq} 0$$

$(\because d, e, f \geq 0 \Rightarrow d+e+f \geq 0)$ and so,

$d.(4r_b r_c - 3bc) + e.(4r_c r_a - 3ca) + f.(4r_a r_b - 3ab) \geq 0 \ \forall \ \Delta$ ABC

with $a \geq b \geq c$ and $0 \leq d \leq e \leq f$, " = " iff Δ ABC is equilateral (QED)