ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $a \geq b \geq c$ and

$0 \leq d \leq e \leq f$ the following relationship holds :

$$d(3bc - 4w_a r_a) + e(3ca - 4w_b r_b) + f(3ab - 4w_c r_c) \geq 0$$

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$$\text{Since } a \geq b \geq c \Rightarrow 3bc \leq 3ca \therefore \text{ in order to prove :}$$

$$3bc - 4w_a r_a \overset{?}{\leq} 3ca - 4w_b r_b, \text{ it suffices to prove :}$$

$$\frac{2ca}{c+a}.\cos\frac{B}{2}.s\tan\frac{B}{2} \overset{?}{\leq} \frac{2bc}{b+c}.\cos\frac{A}{2}.s\tan\frac{A}{2}$$

$$\Leftrightarrow \frac{a^2}{(c+a)^2}.\frac{(s-c)(s-a)}{ca} \overset{?}{\leq} \frac{b^2}{(b+c)^2}.\frac{(s-b)(s-c)}{bc} \text{ and } \because \frac{1}{c+a} \leq \frac{1}{b+c}$$

$$\therefore \text{ it suffices to prove : } a(s-a) \overset{?}{\leq} b(s-b) \Leftrightarrow (a+b)(a-b) \overset{?}{\geq} s(a-b)$$

$$\Leftrightarrow (a-b)(2s-c-s) \overset{?}{\geq} 0 \Leftrightarrow (a-b)(s-c) \overset{?}{\geq} 0 \to \text{true} \because a \geq b$$

$$\therefore 3bc - 4w_a r_a \boxed{\overset{(1)}{\leq}} 3ca - 4w_b r_b \text{ and again, since } a \geq b \geq c \Rightarrow 3ca \leq 3ab$$

$$\therefore \text{ in order to prove : } 3ca - 4w_b r_b \overset{?}{\leq} 3ab - 4w_c r_c, \text{ it suffices to prove :}$$

$$\frac{2ab}{a+b}.\cos\frac{C}{2}.s\tan\frac{C}{2} \overset{?}{\leq} \frac{2ca}{c+a}.\cos\frac{B}{2}.s\tan\frac{B}{2}$$

$$\Leftrightarrow \frac{b^2}{(a+b)^2}.\frac{(s-a)(s-b)}{ab} \overset{?}{\leq} \frac{c^2}{(c+a)^2}.\frac{(s-c)(s-a)}{ca} \text{ and } \because \frac{1}{a+b} \leq \frac{1}{c+a}$$

$$\therefore \text{ it suffices to prove : } b(s-b) \overset{?}{\leq} c(s-c) \Leftrightarrow (b+c)(b-c) \overset{?}{\geq} s(b-c)$$

$$\Leftrightarrow (b-c)(2s-a-s) \overset{?}{\geq} 0 \Leftrightarrow (b-c)(s-a) \overset{?}{\geq} 0 \to \text{true} \because b \geq c$$

$$\therefore 3ca - 4w_b r_b \boxed{\overset{(2)}{\leq}} 3ab - 4w_c r_c \therefore (1) \text{ and } (2) \Rightarrow$$

$$d \leq e \leq f \text{ and } 3bc - 4w_a r_a \leq 3ca - 4w_b r_b \leq 3ab - 4w_c r_c$$

$$\therefore d.(3bc - 4w_a r_a) + e.(3ca - 4w_b r_b) + f.(3ab - 4w_c r_c) \overset{\text{Chebyshev}}{\geq}$$

$$\frac{1}{3}(d+e+f).\sum_{cyc}(3bc - 4w_a r_a) \to (*)$$

$$\text{Now, } 4\sum_{cyc} w_a r_a \overset{\text{AM-GM}}{\leq} 4.\sum_{cyc}\left(\sqrt{s(s-a)}.\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a}\right) \overset{\text{CBS}}{\leq}$$

$$4s.\sqrt{3\sum_{cyc}((s-b)(s-c))} = 4s.\sqrt{3(4Rr+r^2)} \overset{?}{\leq} 3\sum_{cyc} bc = 3(s^2+4Rr+r^2)$$

$$\overset{\text{squaring}}{\Leftrightarrow} 3s^4 - (40Rr + 10r^2)s^2 + 3r^2(4R + r)^2 \boxed{\overset{?}{\underset{(\bullet)}{\geq}}} 0 \text{ and } \because$$

$$3(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (\bullet), \text{it suffices to prove :}$$

$$\text{LHS of } (\bullet) \overset{?}{\geq} 3(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (7R - 5r)s^2 \boxed{\overset{?}{\underset{(\bullet\bullet)}{\geq}}} r(90R^2 - 63Rr + 9r^2);\ \text{and finally}, (7R - 5r)s^2 \overset{\text{Gerretsen}}{\geq}$$

$$(7R - 5r)(16Rr - 5r^2) \overset{?}{\geq} r(90R^2 - 63Rr + 9r^2) \Leftrightarrow 2r(11R - 4r)(R - 2r) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true} \therefore \sum_{\text{cyc}} (3bc - 4w_a r_a) \geq 0 \rightarrow (**) \text{ and } \because$$

$$d, e, f \geq 0 \Rightarrow d + e + f \geq 0 \therefore (*) \text{ and } (**) \Rightarrow$$

$$d.(3bc - 4w_a r_a) + e.(3ca - 4w_b r_b) + f.(3ab - 4w_c r_c) \geq 0\ \forall \Delta ABC$$

$$\text{with } a \geq b \geq c \text{ and } 0 \leq d \leq e \leq f, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$