ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC following relationship holds:

$$2m_a + r_a \le \left(2\sqrt{2}+2\right)R - \left(4\sqrt{2}-5\right)r$$

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$$Let\ x = \sin\frac{A}{2}\ \&\ t = \cos\frac{B-C}{2}\ \&\ \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2x\sqrt{1-x^2}$$

$$now\ 0 < x < 1\ \ and\ \left|\frac{B-C}{2}\right| < \frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}\ \&$$

$$t = \cos\frac{B-C}{2} > \cos\left(\frac{\pi}{2} - \frac{A}{2}\right) = \sin\frac{A}{2} = x$$

$$so\ x < t < 1$$

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\ \&\ A+B+C = \pi$$

$$2\sin\frac{B}{2}\sin\frac{C}{2} = \cos\frac{B-C}{2} - \cos\frac{B+C}{2} = \cos\frac{B-C}{2} - \sin\frac{A}{2} = t - x$$

$$\sin\frac{B}{2}\sin\frac{C}{2} = \frac{t-x}{2}$$

$$r_a = r = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\ \&\ 2\cos\frac{B}{2}\cos\frac{C}{2} = \cos\frac{B+C}{2} + \cos\frac{B-C}{2} =$$

$$= \sin\frac{A}{2} + \cos\frac{B-C}{2} = t + x\ \ so,$$

$$\cos\frac{B}{2}\cos\frac{C}{2} = \frac{(t+x)}{2}$$

$$m_a^2 = \frac{2b^2+2c^2-a^2}{4} = \frac{4R^2(2\sin^2 B + 2\sin^2 C - \sin^2 A)}{4}\ then$$

$$\frac{m_a^2}{R^2} = 2\sin^2 C + 2\sin^2 B - \sin^2 A = 2(1 - \cos(B+C)\cos(B-C))\ - \sin^2 A$$

$$= 2(1 + \cos A\cos(B-C)) - \sin^2 A =$$

$$= 2\left(1 + \left(1 - 2\sin^2\frac{A}{2}\right)\left(2\cos^2\frac{B-C}{2} - 1\right)\right) - \sin^2 A =$$

$$= 2\left(1 - (2x^2-1)(2t^2-1)\right) - 4x^2(1-x^2) = 4t^2 - 8x^2t^2 + 4x^4$$

$$Then\ \frac{2m_a}{R} = 4\sqrt{t^2 - 2x^2t^2 + x^4}, \frac{r_a}{R} = 2x(t+x), \frac{r}{R} = 2x(t-x)$$

Using above result we need to show:

$$2m_a + r_a \le \left(2\sqrt{2} + 2\right)R - \left(4\sqrt{2} - 5\right)r$$

$$\frac{2m_a}{R} + \frac{r_a}{R} + \left(4\sqrt{2} - 5\right)\frac{r}{R} \le \left(2\sqrt{2} + 2\right)$$

$$4\sqrt{t^2 - 2x^2t^2 + x^4} + 2x(t + x) + 2x(t - x)\left(4\sqrt{2} - 5\right) \le \left(2\sqrt{2} + 2\right)$$

$$4\sqrt{(t - x^2)^2} + 2x(t + x) + 2x(t - x)\left(4\sqrt{2} - 5\right) \dot{\le} \left(2\sqrt{2} + 2\right)$$

$$4\sqrt{(1 - x^2)^2} + 2x(1 + x) + 2x(1 - x)\left(4\sqrt{2} - 5\right) \overset{x<t<1}{\le} \left(2\sqrt{2} + 2\right)$$

$$4(1 - x^2) + 2x(1 + x) + 2x(1 - x)\left(4\sqrt{2} - 5\right) \overset{x<t<1}{\le} \left(2\sqrt{2} + 2\right)$$

$$4 + 8\left(\sqrt{2} - 1\right)x(1 - x) - \left(2\sqrt{2} + 2\right) \le 0\ or, \left(\sqrt{2} - 1\right)(2x - 1)^2 \ge 0$$

Equality holds for an equilateral triangle.