ROMANIAN MATHEMATICAL MAGAZINE

In any acute ΔABC following relationship holds:

$$\frac{r_b+r_c}{\sqrt{2(b^2+c^2)}}\ge\cos\frac{A}{2}\ge\frac{2h_a+r_b+r_c}{2(b+c)}$$

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$$(1st\ Part)$$

$$Let\ x=s-a,y=s-b,z=s-c\ then$$

$$a=y+z,b=z+x,c=x+y\ \&\ s=x+y+z$$

$$r_b=\frac{F}{s-b}=\frac{F}{y},r_c=\frac{F}{s-c}=\frac{F}{z}\ \&\ r_b+r_c=\frac{F(y+z)}{yz}$$

$$F^2=s(s-a)(s-b)(s-c)=sxyz$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}=\sqrt{\frac{x(x+y+z)}{(x+z)(x+y)}}$$

$$We\ need\ to\ show:\ \frac{r_b+r_c}{\sqrt{2(b^2+c^2)}}\ge\cos\frac{A}{2}\ or\ \frac{(r_b+r_c)^2}{2(b^2+c^2)}\ge\cos^2\frac{A}{2}$$

$$\frac{\left(\frac{F(y+z)}{yz}\right)^2}{2((x+z)^2+(x+y)^2)}\ge\frac{x(x+y+z)}{(x+z)(x+y)}$$

$$\frac{\frac{sxyz(y+z)^2}{(yz)^2}}{2((x+z)^2+(x+y)^2)}\ge\frac{x(x+y+z)}{(x+z)(x+y)}$$

$$(y+z)^2(x+z)(x+y)\ge 2yz((x+z)^2+(x+y)^2)$$

$$x^2y^2+x^2z^2-2x^2yz+xy^3-xy^2z-xyz^2+xz^3-y^3z+3y^2z^2-yz^3\ge 0$$

$$x^2(y-z)^2+x(\acute{y}-z)^2(y+z)-yz(y-z)^2\ge 0$$

$$(y-z)^2(x^2+x(y+z)-yz)\ge 0\ true$$

$$\left(\begin{array}{c}as\ triangle\ is\ acute\ then\ a^2<b^2+c^2 or\\ (y+z)^2<(x+z)^2+(x+y)^2 or, x^2+x(y+z)-yz>0\end{array}\right)$$

$$(2nd\ Part)$$

$$acute\ triangle, a^2+c^2>b^2\ or$$

$$(y+x)^2+(z+y)^2>(x+z)^2\ or, xy+y^2+yz-xz>0$$

$$a^2+b^2>c^2\ or,(y+z)^2+(z+x)^2>(x+y)^2 or, xz+yz+z^2-xy>0$$

$$using\ above\ we\ get\ (x(y-z)+y^2+3yz)=(xy+yz+y^2-xz)+2yz>0\ (A)$$

$$and$$

$$(x(y-z)-3yz-z^2))=xy-yz-3yz-z^2=-(xz+z^2+yz-xy)-2yz<0(B)$$

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$$\frac{2h_a + r_b + r_c}{2(b+c)} = \frac{\frac{4F}{a} + \frac{F}{s-b} + \frac{F}{s-c}}{2(b+c)} = \frac{F\left(\frac{4}{y+z} + \frac{1}{y} + \frac{1}{z}\right)}{2(2x+y+z)} = \frac{F(y^2+6yz+z^2)}{2yz(y+z)(2x+y+z)}$$

$$We\ need\ to\ show:$$

$$\frac{2h_a + r_b + r_c}{2(b+c)} \le \cos\frac{A}{2}$$

$$\frac{F(y^2+6yz+z^2)}{2yz(y+z)(2x+y+z)} \le \sqrt{\frac{x(x+y+z)}{(x+z)(x+y)}}$$

$$\frac{sxyz(y^2+6yz+z^2)^2}{4y^2z^2(y+z)^2(2x+y+z)^2} \le \frac{x(x+y+z)}{(x+z)(x+y)}$$

$$4yz(y+z)^2(2x+y+z)^2 - (x+y)(x+z)(y^2+6yz+z^2) \le 0$$

$$-(y-z)^2(x(y-z)+y^2+3yz)(x(y-z)-3yz-z^2)) \ge 0 \ \ true\ by\ (A)\ \&\ (B)$$

Equality holds for an equilateral triangle.