ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$8m_bm_c \ge max\left\{3a^2 - 12Rr + 60r^2, 4a^2 + b^2 + c^2 - \frac{9b^2c^2(b-c)^2}{16F^2}\right\}$$

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$$(m_b - m_c)^2 \le \frac{s^2 + 2Rr - 31r^2}{2}$$

(Reference : Inequality in Triangle – 345 by Dang Ngoc Minh; published at www.ssmrmh.ro)

$$\Rightarrow 4m_b^2 + 4m_c^2 - 8m_bm_c \le 2s^2 + 4Rr - 62r^2$$

$$\Rightarrow 8m_bm_c \ge 2c^2 + 2a^2 - b^2 + 2a^2 + 2b^2 - c^2 - 2s^2 - 4Rr + 62r^2$$

$$= 4a^2 + b^2 + c^2 - 2s^2 - 4Rr + 62r^2 = 3a^2 + \sum_{cyc} a^2 - 2s^2 - 4Rr + 62r^2$$

$$= 3a^2 + 2(s^2 - 4Rr - r^2) - 2s^2 - 4Rr + 62r^2 = 3a^2 - 12Rr + 60r^2$$

$$\therefore \boxed{8m_bm_c \ge 3a^2 - 12Rr + 60r^2, " = " \text{ iff } a = b = c} \text{ and again,}$$

$$8m_bm_c \overset{?}{\ge} 4a^2 + b^2 + c^2 - \frac{9b^2c^2(b-c)^2}{16F^2}$$

$$\Leftrightarrow 8m_bm_c \overset{?}{\ge} 4m_b^2 + 4m_c^2 - \frac{9b^2c^2(b-c)^2}{16F^2} \Leftrightarrow \frac{9b^2c^2(b-c)^2}{16F^2} \overset{?}{\ge} 4(m_b - m_c)^2$$

$$= \frac{4(m_b^2 - m_c^2)^2}{(m_b + m_c)^2} = \frac{\left((2c^2 + 2a^2 - b^2) - (2a^2 + 2b^2 - c^2)\right)^2}{4(m_b + m_c)^2} = \frac{9(b-c)^2(b+c)^2}{4(m_b + m_c)^2}$$

$$\Leftrightarrow \frac{bc}{2F}.(m_b + m_c) \overset{?}{\ge} b + c \left(\because (b-c)^2 \overset{?}{\ge} 0\right) \to \text{true}$$

$$\because \frac{bc}{2F}.(m_b + m_c) \ge \frac{bc}{2F}.(h_b + h_c) = \frac{abc(b+c)}{2F.2R} = \frac{4RF(b+c)}{4RF} = b + c$$

$$\therefore \boxed{8m_bm_c \ge 4a^2 + b^2 + c^2 - \frac{9b^2c^2(b-c)^2}{16F^2}, " = " \text{ iff } a = b = c} \text{ and so,}$$

$$8m_bm_c \ge max\left\{3a^2 - 12Rr + 60r^2, 4a^2 + b^2 + c^2 - \frac{9b^2c^2(b-c)^2}{16F^2}\right\} \forall \text{ ABC},$$

$$" = " \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$