ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$m_a h_b h_c + m_b h_c h_a + m_c h_a h_b \le 3 r_a r_b r_c$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$4m_a\sqrt{a^2+b^2+c^2} = 2\sqrt{2b^2+2c^2-a^2}\sqrt{a^2+b^2+c^2} \overset{AM-GM}{\le} 3(b^2+c^2)$$

$$m_a \le \frac{3(b^2+c^2)}{4\sqrt{a^2+b^2+c^2}} \, (1)$$

$$m_a h_b h_c + m_b h_c h_a + m_c h_a h_b = 4F^2\left(\frac{m_a}{bc}+\frac{m_b}{ca}+\frac{m_c}{ab}\right) = \frac{4F^2}{abc}(am_a+bm_b+cm_c) =$$

$$= \frac{F}{R}(am_a+bm_b+cm_c) \overset{(1)}{\le} \frac{F}{R}\sum \frac{3a(b^2+c^2)}{4\sqrt{a^2+b^2+c^2}} = \frac{F}{4R}\frac{3}{\sqrt{a^2+b^2+c^2}}\sum a(b^2+c^2) \le$$

$$\overset{CBS}{\le} \frac{F}{R}\frac{3}{4\sqrt{a^2+b^2+c^2}} \cdot \frac{1}{3}\sum a \cdot \left(\sum b^2+c^2\right) = \frac{F}{R}\frac{3}{4\sqrt{a^2+b^2+c^2}} \cdot \frac{1}{3}\sum a \cdot \left(2\sum b^2\right) =$$

$$= \frac{F}{4R} 2s \cdot 2\sqrt{a^2+b^2+c^2} \overset{Leibniz}{\le} \frac{F}{4R} 4s \cdot \sqrt{9R^2} = 3Rs \cdot \frac{F}{R} = 3Fs = 3r_a r_b r_c.$$

Equality holds for an equilateral triangle.