ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$(m_a+m_b+m_c)^4 \le 16\sum_{cyc} b^2c^2 - \frac{117a^2b^2c^2}{16\sum_{cyc} a^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We seek to prove : } \frac{1}{16}\left(\sum_{cyc} a\right)^4 \overset{?}{\le} \frac{16}{9}\sum_{cyc} a^2b^2 - \frac{52}{27}\cdot\frac{m_a^2m_b^2m_c^2}{\sum_{cyc} a^2} \text{ and since :}$$

$$m_a^2m_b^2m_c^2 = \frac{s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3}{16}$$

(Reference : Solution to Inequality in Triangle – 124 by Dang Ngoc Minh; published at www.ssmrmh.ro)

$$\therefore \text{ the inequality becomes : } s^4 \overset{?}{\le} \frac{16}{9}\left((s^2+4Rr+r^2)^2 - 16Rrs^2\right) - \frac{52}{27}\cdot\frac{s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3}{32(s^2-4Rr-r^2)}$$

$$\Leftrightarrow 155s^6 - (3588Rr-171r^2)s^4 + r^2(19212R^2+4632Rr+45r^2)s^2 - r^3(23744R^3+17808R^2r+4452Rr^2+371r^3) \overset{?}{\underset{(1)}{\ge}} 0 \text{ and } \because (s^2-16Rr+5r^2)^3$$

$$\overset{\text{Gerretsen}}{\ge} 0 \therefore \text{ to prove (1), it suffices to prove : LHS of (1)} \overset{?}{\ge} 155(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (642R-359r)s^4 - r(16638R^2-13172Rr+1930r^2)s^2 + r^2(32(3183R^3-3193R^2r+945Rr^2-103r^3)+8R^2r+18Rr^2+5r^3) \overset{?}{\underset{(2)}{\ge}} 0 \text{ and}$$

$$\because (642R-359r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{ to prove (2), suffices to prove :}$$

$$\text{LHS of (2)} \overset{?}{\ge} (642R-359r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (1953R^2-2368Rr+830r^2)s^2 \overset{?}{\underset{(3)}{\ge}} r(31248R^3-46228R^2r+21616Rr^2-2842r^3)$$

$$\text{Finally, } (1953R^2-2368Rr+830r^2)s^2 \overset{\text{Rouche + Euler}}{\ge} (1953R^2-2368Rr+830r^2)\left(16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\right) \overset{?}{\ge} \text{RHS of (3)}$$

$$\Leftrightarrow 528t^3 - 1345t^2 + 754t - 352 \overset{?}{\ge} 0 \quad \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(528t^2-289t+176) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \text{ is true}$$

$$\therefore \frac{1}{16}\left(\sum_{cyc} a\right)^4 \leq \frac{16}{9}\cdot\sum_{cyc} a^2b^2 - \frac{52}{27}\cdot\frac{m_a^2m_b^2m_c^2}{\sum_{cyc} a^2}$$ and implementing it on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$, whose medians as a consequence of trivial calculations are $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ respectively, we arrive at :

$$\frac{1}{16}\cdot\frac{16}{81}\left(\sum_{cyc} m_a\right)^4 \leq \frac{16}{9}\cdot\frac{16}{81}\cdot\sum_{cyc} m_a^2m_b^2 - \frac{52}{27}\cdot\frac{1}{64}\cdot\frac{a^2b^2c^2}{\frac{4}{9}\cdot\sum_{cyc} m_a^2}$$

$$= \frac{16}{9}\cdot\frac{16}{81}\cdot\frac{9}{16}\cdot\sum_{cyc} a^2b^2 - \frac{52}{27}\cdot\frac{1}{64}\cdot\frac{a^2b^2c^2}{\frac{4}{9}\cdot\frac{3}{4}\cdot\sum_{cyc} a^2}$$

$$\Rightarrow (m_a + m_b + m_c)^4 \leq 16\sum_{cyc} b^2c^2 - \frac{117a^2b^2c^2}{16\sum_{cyc} a^2}\ \forall\ ABC,$$

$'' = ''$ iff Δ ABC is equilateral (QED)