ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\max\left\{\frac{1}{r_a+r_b}+\frac{1}{r_a+r_c},\frac{1}{2r_a}+\frac{1}{r_b+r_c}\right\}\le\frac{2}{h_b+h_c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x=s-a,y=s-b,z=s-c$ and then : $a=y+z,b=z+x,$ $c=x+y$ and $s=x+y+z$ and now : $\frac{1}{r_a+r_b}+\frac{1}{r_a+r_c}\overset{?}{\le}\frac{2}{h_b+h_c}$

$$\Leftrightarrow\frac{(s-a)(s-b)}{rsc}+\frac{(s-a)(s-c)}{rsb}\overset{?}{\le}\frac{2bc}{2rs(b+c)}$$

$$\Leftrightarrow(z+x)^2(x+y)^2\overset{?}{\ge}\big(xy(z+x)+zx(x+y)\big)(2x+y+z)$$

$$\overset{expanding\ and\ simplifying}{\Leftrightarrow}x^4-2x^2yz+y^2z^2\overset{?}{\ge}0\Leftrightarrow(x^2-yz)^2\overset{?}{\ge}0\to\text{true}$$

and again, $\frac{1}{2r_a}+\frac{1}{r_b+r_c}\overset{?}{\le}\frac{2}{h_b+h_c}\Leftrightarrow\frac{s-a}{2rs}+\frac{(s-b)(s-c)}{rsa}\overset{?}{\le}\frac{2bc}{2rs(b+c)}$

$$\Leftrightarrow(2x+y+z)(x(y+z)+2yz)\overset{?}{\ge}2(y+z)(z+x)(x+y)$$

$$\overset{expanding\ and\ simplifying}{\Leftrightarrow}x(y^2-2yz+z^2)\overset{?}{\ge}0\Leftrightarrow x(y-z)^2\overset{?}{\ge}0\to\text{true}$$

$$\therefore\frac{1}{r_a+r_b}+\frac{1}{r_a+r_c},\frac{1}{2r_a}+\frac{1}{r_b+r_c}\le\frac{2}{h_b+h_c}$$

$$\Rightarrow\max\left\{\frac{1}{r_a+r_b}+\frac{1}{r_a+r_c},\frac{1}{2r_a}+\frac{1}{r_b+r_c}\right\}\le\frac{2}{h_b+h_c}\ \forall\ \text{ABC},$$

$''=''$ iff Δ ABC is equilateral (QED)