ROMANIAN MATHEMATICAL MAGAZINE

In $a$ny Δ ABC the fol$l$owing relationship ho$l$ds :

$$\frac{21}{16}\cdot\sum_{cyc}b^2c^2+8.\frac{m_a^2m_b^2m_c^2}{\sum_{cyc}a^2}\geq(m_am_b+m_bm_c+m_cm_a)^2\geq\frac{9}{32}\cdot\sum_{cyc}a^4+30.\frac{m_a^2m_b^2m_c^2}{\sum_{cyc}a^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, we seek to prove : }2\sum_{cyc}a^2b^2+\frac{9a^2b^2c^2}{\sum_{cyc}a^2}\overset{?}{\geq}3abc\left(\sum_{cyc}a\right)$$

$$\Leftrightarrow 2((s^2+4Rr+r^2)^2-16Rrs^2)+\frac{72R^2r^2s^2}{s^2-4Rr-r^2}\overset{?}{\geq}24Rrs^2$$

$$\Leftrightarrow s^6-(24Rr-r^2)s^4+r^2(132R^2+20Rr-r^2)s^2-r^3(4R+r)^3\boxed{\underbrace{\overset{?}{\geq}}_{①}}0\ and\ \because$$

$$(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0\ \therefore\text{ in order to prove ①, it suffices to prove :}$$

$$\text{LHS of ①}\overset{?}{\geq}(s^2-16Rr+5r^2)^3\Leftrightarrow(12R-7r)s^4-$$

$$r(318R^2-250Rr+38r^2)s^2+r^2(2016R^3-1944R^2r+594Rr^2-63r^3)\boxed{\underbrace{\overset{?}{\geq}}_{②}}0$$

$$and\ \because P=(12R-7r)\left((s^2-16Rr+5r^2)(R-r)-r^2(R-2r)\right)^2\overset{\text{Rouche + Euler}}{\geq}0$$

$$\therefore\text{ in order to prove ②, it suffices to prove : }(R-r)^2.\text{LHS of ②}\overset{?}{\geq}P$$

$$\Leftrightarrow(66R^4-202R^3r+200R^2r^2-68Rr^3+4r^4)s^2\boxed{\underbrace{\overset{?}{\geq}}_{③}}$$

$$r\left(1056R^5-3496R^4r+3934R^3r^2-1717R^2r^3+228Rr^4\right)$$

$$\text{Now}\because 66R^4-202R^3r+200R^2r^2-68Rr^3+4r^4=$$

$$(R-2r)(66R^3-70R^2r+60Rr^2+52r^3)+108r^3\overset{\text{Euler}}{\geq}108r^3>0\ \therefore\text{ LHS of ③}$$

$$\overset{\text{Rouche + Euler}}{\geq}(66R^4-202R^3r+200R^2r^2-68Rr^3+4r^4)\left(\frac{16Rr-5r^2+r^2(R-2r)}{R-r}\right)$$

$$\overset{?}{\geq}\text{RHS of ③}\Leftrightarrow 8t^4-43t^3+79t^2-56t+12\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)^2(8t^2-11t+3)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow③\Rightarrow②\Rightarrow①\text{ is true}$$

$$\therefore 2\sum_{cyc}a^2b^2+\frac{9a^2b^2c^2}{\sum_{cyc}a^2}\geq 3abc\left(\sum_{cyc}a\right);\text{ implementing it on }a\ \Delta\text{ XYZ with sides}$$

$$\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3},\text{ whose medians \& }area\ are\ \frac{a}{2},\frac{b}{2},\frac{c}{2}\ and\ \frac{F}{3}\text{ respectively, we get :}$$

$$2.\frac{16}{81}.\sum_{\text{cyc}} m_a^2m_b^2+9.\frac{\frac{64}{729}.m_a^2m_b^2m_c^2}{\frac{4}{9}.\sum_{\text{cyc}} m_a^2}\geq 3.\frac{8}{27}m_am_bm_c.\frac{2}{3}.\left(\sum_{\text{cyc}} m_a\right)$$

$$\Rightarrow 2.\frac{16}{81}.\frac{9}{16}.\sum_{\text{cyc}} b^2c^2+9.\frac{\frac{64}{729}.m_a^2m_b^2m_c^2}{\frac{4}{9}.\frac{3}{4}.\sum_{\text{cyc}} a^2}\geq 3.\frac{8}{27}m_am_bm_c.\frac{2}{3}.\left(\sum_{\text{cyc}} m_a\right)$$

$$\Rightarrow \frac{2}{9}.\sum_{\text{cyc}} b^2c^2+\frac{8}{27}.\frac{9}{16}.\sum_{\text{cyc}} b^2c^2+\frac{64}{27}.\frac{m_a^2m_b^2m_c^2}{\sum_{\text{cyc}} a^2}\geq$$

$$\frac{8}{27}\left(2m_am_bm_c\left(\sum_{\text{cyc}} m_a\right)+\sum_{\text{cyc}} m_a^2m_b^2\right)$$

$$\Rightarrow \boxed{\frac{21}{16}.\sum_{\text{cyc}} b^2c^2+8.\frac{m_a^2m_b^2m_c^2}{\sum_{\text{cyc}} a^2}\geq (m_am_b+m_bm_c+m_cm_a)^2}$$

We now seek to prove : $\left(\sum_{\text{cyc}} ab\right)^2\overset{?}{\geq}\sum_{\text{cyc}} a^2b^2+\frac{45a^2b^2c^2}{32\sum_{\text{cyc}} a^2}-\frac{F^2}{2}\overset{?}{\geq}0\Leftrightarrow$

$(4R+2r)\boxed{\underbrace{\overset{?}{\geq}}_{\textcircled{4}}}r(61R^2+12Rr+2r^2)$; & $\text{LHS}_{\textcircled{4}}\overset{\text{Gerretsen}}{\geq}\geq(4R+2r)(16Rr-5r^2)$

$\overset{?}{\geq}$ RHS of ④ $\Leftrightarrow R^2-4r^2\overset{?}{\geq}0\rightarrow$ true via Euler $\therefore\frac{1}{16}.\left(\sum_{\text{cyc}} ab\right)^2\geq$

$\frac{1}{16}.\sum_{\text{cyc}} a^2b^2+\frac{45a^2b^2c^2}{32\sum_{\text{cyc}} a^2}-\frac{F^2}{2}$ and implementing it on a triangle XYZ, we get :

$$\frac{1}{16}.\frac{16}{81}\left(\sum_{\text{cyc}} m_am_b\right)^2\geq\frac{1}{16}.\frac{16}{81}.\frac{9}{32}\left(\sum_{\text{cyc}} a^4+16F^2\right)+\frac{45.\frac{64}{729}.m_a^2m_b^2m_c^2}{32.\frac{4}{9}.\frac{3}{4}.\sum_{\text{cyc}} a^2}-\frac{F^2}{18}\Rightarrow$$

$$\boxed{(m_am_b+m_bm_c+m_cm_a)^2\geq\frac{9}{32}.\sum_{\text{cyc}} a^4+30.\frac{m_a^2m_b^2m_c^2}{\sum_{\text{cyc}} a^2}}\text{ '' = '' iff } a=b=c\text{ (QED)}$$