ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{1}{h_a+r_b}+\frac{1}{h_a+r_c}\ge\frac{1}{2h_a}+\frac{1}{r_b+r_c}\ge\frac{1}{\sqrt{r_br_c}}$$

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$$\frac{1}{h_a+r_b}+\frac{1}{h_a+r_c}\overset{?}{\ge}\frac{1}{2h_a}+\frac{1}{r_b+r_c}$$

$$\Leftrightarrow\frac{a(s-b)}{rs\big(a+2(s-b)\big)}+\frac{a(s-c)}{rs\big(a+2(s-c)\big)}\overset{?}{\ge}\frac{a}{4rs}+\frac{(s-b)(s-c)}{rsa}$$

$$\Leftrightarrow\frac{a^2\big((s-b)+(s-c)\big)+4a(s-b)(s-c)}{\big(a+2(s-b)\big)\big(a+2(s-c)\big)}\overset{?}{\ge}\frac{a^2+4(s-b)(s-c)}{4a}$$

$$\Leftrightarrow\frac{a^3+4a(s-b)(s-c)}{\big(a+2(s-b)\big)\big(a+2(s-c)\big)}\overset{?}{\ge}\frac{a^2+4(s-b)(s-c)}{4a}$$

$$\Leftrightarrow 4a^2\overset{?}{\ge}a^2+4(s-b)(s-c)+2a\big((s-b)+(s-c)\big)$$

$$\Leftrightarrow 4a^2\overset{?}{\ge}a^2+4(s-b)(s-c)+2a^2\Leftrightarrow a^2\overset{?}{\ge}a^2-(b-c)^2\Leftrightarrow(b-c)^2\overset{?}{\ge}0$$

$$\to\text{true and again, }\frac{1}{2h_a}+\frac{1}{r_b+r_c}\overset{?}{\ge}\frac{1}{\sqrt{r_br_c}}$$

$$\Leftrightarrow\frac{a^2+4(s-b)(s-c)}{4a.rs}\overset{?}{\ge}\frac{\sqrt{(s-b)(s-c)}}{rs}$$

$$\Leftrightarrow a^2-4a.\sqrt{(s-b)(s-c)}+4(s-b)(s-c)\overset{?}{\ge}0$$

$$\Leftrightarrow\left(a-2\sqrt{(s-b)(s-c)}\right)^2\overset{?}{\ge}0\to\text{true}$$

$$\therefore\frac{1}{h_a+r_b}+\frac{1}{h_a+r_c}\ge\frac{1}{2h_a}+\frac{1}{r_b+r_c}\ge\frac{1}{\sqrt{r_br_c}}\ \forall\,ABC,''=''\text{ iff }b=c\ (QED)$$