ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$r_a w_b w_c + r_b w_c w_a + r_c w_a w_b \ge \frac{9\sqrt{3}}{8}.abc$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS} = \frac{16Rr^2s^2}{s^2+2Rr+r^2}.\sum_{\text{cyc}}\frac{r_a}{w_a} \ge \frac{16Rr^2s^2}{s^2+2Rr+r^2}.\frac{29R^2-2s^2+10Rr+14r^2}{16Rr}$$

$$\left(\begin{array}{c}\text{Reference : Inequality in Triangle} - 228 \text{ by Dang Ngoc Minh;}\\ \text{publshed at www.ssmrmh.ro}\end{array}\right) \overset{?}{\ge} \frac{9\sqrt{3}}{8}.abc$$

$$= \frac{9\sqrt{3}.Rrs}{2} \overset{\text{squaring}}{\Leftrightarrow} 16s^6 - (707R^2+160Rr+224r^2)s^4 +$$

$$(3364R^4+1348R^3r+3162R^2r^2+1120Rr^3+784r^4)s^2 -$$

$$243R^2r^2(2R+r)^2 \boxed{\underset{①}{\overset{?}{\ge}}} 0 \text{ and } \because (s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{ to prove } ①,$$

$$\text{it suffices to prove : LHS of } ① \overset{?}{\ge} 16(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow -(707R^2-608Rr+464r^2)s^4$$

$$+(3364R^4+1348R^3r-9126R^2r^2+8800Rr^3-416r^4)s^2 -$$

$$r^2(972R^4-64564R^3r+61683R^2r^2-19200Rr^3+2000r^4) \boxed{\underset{②}{\overset{?}{\ge}}} 0$$

$$\text{Now, Rouche} \Rightarrow s^2-(m-n) \ge 0 \text{ and } s^2-(m+n) \le 0, \text{ where } m =$$

$$2R^2+10Rr-r^2 \;\&\; n = 2(R-2r)\sqrt{R^2-2Rr} \therefore \big(s^2-(m+n)\big)\big(s^2-(m-n)\big) \le 0$$

$$\Rightarrow s^4 - s^2(2m)+m^2-n^2 \le 0 \Rightarrow s^4 - s^2(4R^2+20Rr-2r^2)+r(4R+r)^3 \le 0$$

$$\therefore \text{ in order to prove } ②, \text{ it suffices to prove : LHS of } ② \overset{?}{\ge}$$

$$-(707R^2-608Rr+464r^2)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$

$$\Leftrightarrow (134R^4-2590R^3r+648R^2r^2-424Rr^3+128r^4)s^2 +$$

$$r\big(11312R^5-1487R^4r+18390R^3r^2-11500R^2r^3+6040Rr^4-384r^5\big) \boxed{\underset{③}{\overset{?}{\ge}}} 0$$

$$\text{and } \because R \overset{\text{Euler}}{\ge} 2r \therefore ③ \text{ is trivially true if :}$$

$$134R^4-2590R^3r+648R^2r^2-424Rr^3+128r^4 \ge 0 \text{ and if :}$$

$$134R^4-2590R^3r+648R^2r^2-424Rr^3+128r^4 < 0, \text{ then LHS of } ③ \overset{\text{Gerretsen}}{\ge}$$

$$(134R^4-2590R^3r+648R^2r^2-424Rr^3+128r^4)(4R^2+4Rr+3r^2) +$$

$$r\big(11312R^5-1487R^4r+18390R^3r^2-11500R^2r^3+6040Rr^4-384r^5\big) \overset{?}{\ge} 0$$

$$\Leftrightarrow 536t^5 + 1488t^4 - 8853t^3 + 11516t^2 - 10740t + 5280 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(536t^4 + 2560t^3 - 3733t^2 + 4050t - 2640) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \text{ is true } \forall \Delta ABC \,\&\, \text{so}, r_a w_b w_c + r_b w_c w_a + r_c w_a w_b \geq \frac{9\sqrt{3}}{8}.abc$$

$\forall$ ABC, '' = '' iff Δ ABC is equilateral (QED)